FUNCTIONS AND HOW THEY CHANGE

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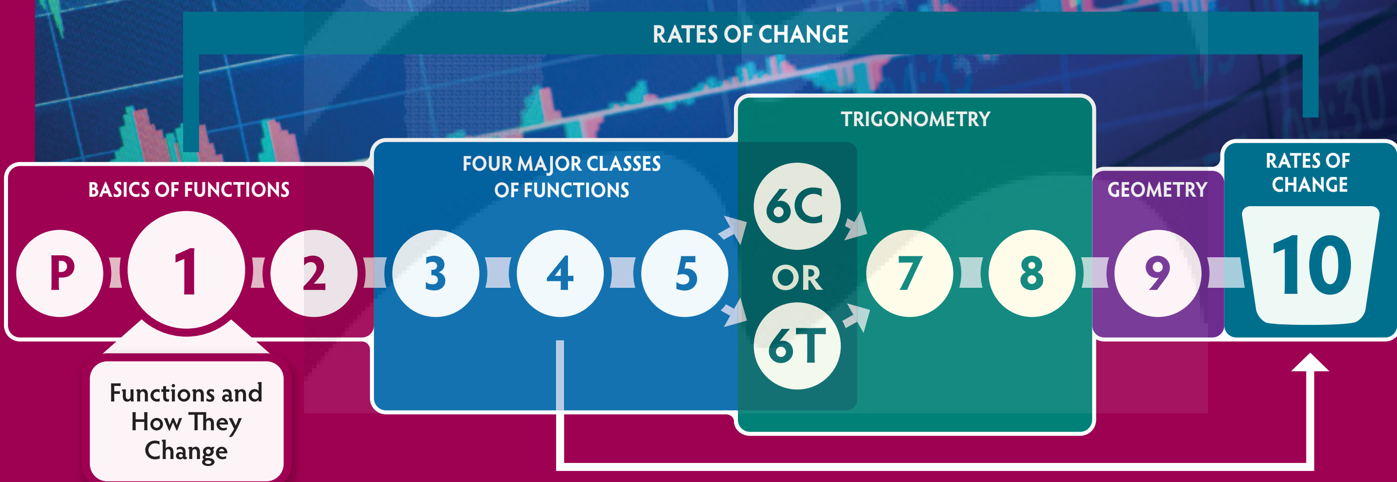

This chapter presents the basics of functions and how they change, foreshadowing the in-depth exploration of the major classes of functions that will come in later chapters. You will learn about the rate of change of a function, the shape of a graph, and the limiting behavior of functions.

1.1 The Basics of Functions
1.2 Average Rates of Change
1.3 Graphs and Rates of Change
1.4 Limits and End Behavior of Graphs

On a very basic level, calculus is simply the study of functions and how they change. A function is a description of how one quantity depends on another—for example, how our weekly pay depends on the number of hours we work. We can use graphs to visualize these relationships. A graph that may receive more scrutiny than any other is associated with the Dow Jones Industrial Average (**Figure 1.1**), which illustrates how the value of stocks depends on time.

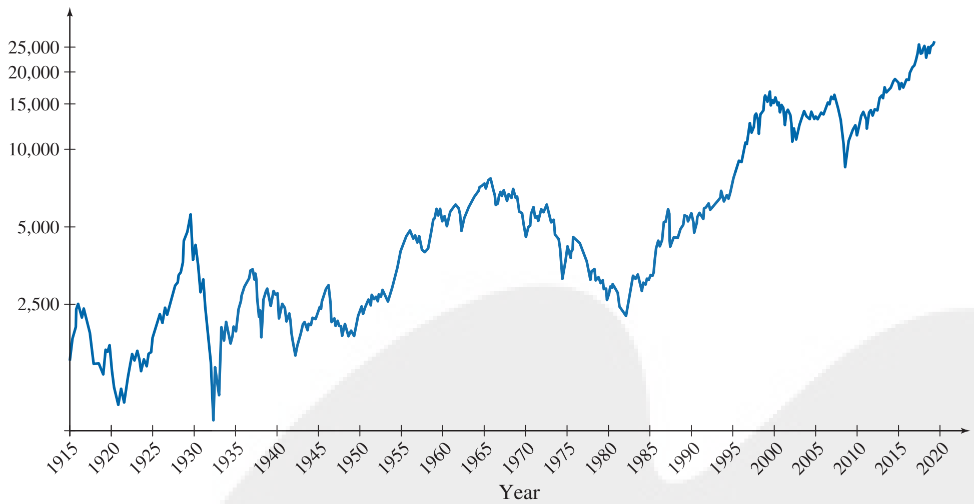

Figure 1.1 Dow-Jones averages

This graph shows a change in relationship (in other words, whether stocks are increasing or decreasing in value). Understanding how the market changes is the key to profitable, short-term investments. But investment counselors advise a focus on the long-term behavior of the market rather than on its daily fluctuations. When the graph is viewed over the long term, it is possible to see that the Dow Jones Industrial Average (an average value of 30 leading U.S. stocks) shows a steady increase over many years. Because of this long-term view, the stock market has been a cornerstone of many long-term investment plans such as retirement accounts. The importance of a dynamic view (contrasted with a static view) of the stock market carries over to the study of virtually all functions and their corresponding graphs. Functions, graphs, rates of change, and long-term behavior are central themes of calculus, of this chapter, and of this book.

1.1 The Basics of Functions

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1.2 Average Rates of Change
1.3 Graphs and Rates of Change
1.4 Limits and End Behavior of Graphs

Functions are fundamental tools for modeling real-world phenomena.

In this section, you will learn to:

1. Evaluate functions given by formulas, words, or tables of values.
2. Interpret functional notation.
3. Identify the domain and range of a given function.
4. Determine whether a given equation defines a function.
5. Determine whether a given graph defines a function.
6. Solve applied problems using the concept of a function.

Definition of a Function

A function gives a rule for assigning some objects to others.

A **function** is a rule that assigns to each object in one collection *exactly one* object in another collection.

The **domain** of a function is the collection of objects for which this rule of assignment is defined.

The term *function* is common in English usage. For example, we may say that weight is a function of calorie intake to indicate that one's weight depends on one's intake of calories. Similarly, we may say that profit is a function of items sold, meaning that our profit is determined by the number of items sold. Earlier we defined functions informally by saying they describe how one quantity depends on another. To prepare for the use of functions in calculus, we need to give a more formal definition of a **function** as a rule assigning to each object in one collection, the **domain**, an object in another collection.

The rules of assignment we will encounter are usually given by a formula whose domain consists of numbers. However, a function might also assign passwords to people, or prices to TV sets, or desks to dorm rooms.

The formula $y = 2x$ determines y as a function of x because to each number x we assign exactly one number y, namely $y = 2x$. In this setting, we refer to x as the function input and the resulting value $y = 2x$ as the function output. The domain of a function is the collection of all allowed inputs.

On the other hand, the formula $y = \pm\sqrt{x}$ does not determine y as a function of x because it assigns to a number x both $\sqrt{x}$ and $-\sqrt{x}$. (For example, the formula assigns to $x = 4$ both $y = 2$ and $y = -2$. Thus, the input $x = 4$ gives two outputs, $y = 2$ and $y = -2$.) A function cannot assign two outputs to any single input.

A classic representation of a function as the process that produces the output from the input is shown in Figure 1.2.

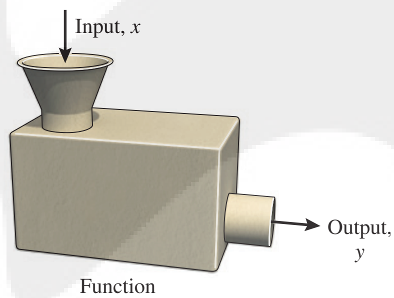


Figure 1.2 An illustration of a function: A function is the process that produces outputs from given inputs.

Evaluating Functions

Functions are evaluated by replacing the independent variable by the desired quantity.

A typical example of a function given by a formula is $y = x^2 + 1$, and we say "y is a function of x." If we are given a value for x, the equation tells us how to find the resulting value of y. For example, if $x = 4$, then putting 4 in place of x in the equation gives $y = 4^2 + 1 = 17$. Because the value of y depends on what value is assigned to x, we call y the dependent variable, and we call x the independent variable.

To emphasize the relationship, the notation $f(x)=x^2+1$ is commonly used in place of $y=x^2+1$. The expression $f(x)$ is referred to as functional notation and is read "f of x." The name of the function is f, and $f(x)$ represents the value of the function given the independent variable x. In the preceding example where $x=4$ resulted in $y=17$, we would say that $f(4)=17$.

This idea holds even for more complicated expressions, as **Table 1.1** shows.

Table 1.1 Finding the Value of a Function

For $f(x)=x^2+1$

Functional notation	Meaning	Method of calculation	Result
$f(5)$	f evaluated at 5	Replace x by 5	$f(5)=5^2+1=26$
$f(y)$	f evaluated at y	Replace x by y	$f(y)=y^2+1$
$f(x+2)$	f evaluated at $x+2$	Replace x by $x+2$	$f(x+2)=(x+2)^2+1=x^2+4x+5$
$f(x^2)$	f evaluated at x^2	Replace x by x^2	$f(x^2)=(x^2)^2+1=x^4+1$
f(Apple)	f evaluated at Apple	Replace x by Apple	$f(\text{Apple})=\text{Apple}^2+1$

EXAMPLE 1.1 Finding Function Values

a. If $f(x)=x^2-x-1$, find the value of $f(4)$.

b. If $h(s)=\dfrac{s}{s+1}$, find and simplify $h(s+1)$.

SOLUTION

a. To find the value of $f(4)$, we replace each occurrence of x by 4 in the formula $f(x)=x^2-x-1$:

$$f(x)=x^2-x-1 \quad \blacktriangleleft \text{ Use the function definition.}$$

$$f(4)=4^2-4-1 \quad \blacktriangleleft \text{ Replace } x \text{ by } 4.$$

$$f(4)=11 \quad \blacktriangleleft \text{ Simplify.}$$

b. To find $h(s+1)$, we replace each occurrence of s by $s+1$ in the formula:

$$h(s)=\frac{s}{s+1} \quad \blacktriangleleft \text{ Use the function definition.}$$

$$h(s+1)=\frac{(s+1)}{(s+1)+1} \quad \blacktriangleleft \text{ Replace each occurrence of } s \text{ by } s+1.$$

$$=\frac{s+1}{s+2} \quad \blacktriangleleft \text{ Simplify.}$$

TRY IT YOURSELF 1.1 Brief answers provided at the end of the section.

a. If $g(x)=4x+\sqrt{x}$, find the value of $g(9)$.

b. If $h(t)=t^2-1$, find and simplify $h(t-1)$.

Functions are often defined by formulas. But graphs, tables of values, and verbal descriptions are also common presentations of functions.

EXAMPLE 1.2 Alternative Function Presentations

a. A technician measures the voltage $V(t)$, in volts, of a battery that is being charged. Here t is the time, in minutes, since the process of charging began. The results are in the table. Find the value of $V(20)$, and explain what it means.

b. If $G(n)$ is the grade you receive on math exam number n, explain the meaning of $G(2)$ in terms of your math exams.

t	$V(t)$
0	3.4
20	3.9
40	3.9
60	4.0

SOLUTION

a. To find the value of $V(20)$, we locate the row of the table corresponding to $t = 20$ and look in the second column. We see that $V(20) = 3.9$ volts. This says that the voltage measured after 20 minutes is 3.9 volts.

b. The expression $G(2)$ means the grade you receive on the second math exam.

TRY IT YOURSELF 1.2 Brief answers provided at the end of the section.

Use the table in the example to find the value of $V(60)$, and explain what it means.

EXAMPLE 1.3 Piecewise-Defined Functions

Sometimes functions are defined by different formulas for different x-values. These are called piecewise-defined functions. For example, consider the function f defined by

$$f(x) = \begin{cases} 2x & \text{if } x < 0 \\ -5x & \text{if } x \geq 0 \end{cases}$$

Calculate the values of $f(-3)$ and $f(3)$.

SOLUTION

Because $-3 < 0$, to calculate $f(-3)$ we use the part of the function definition that says $f(x) = 2x$. This gives $f(-3) = 2 \times (-3) = -6$. Because $3 \geq 0$, to calculate $f(3)$ we use the part of the definition that says $f(x) = -5x$. Thus, $f(3) = -5 \times 3 = -15$.

TRY IT YOURSELF 1.3 Brief answers provided at the end of the section.

Find the value of $f(-1)$.

Domain and Range

The domain comprises the allowable inputs for a function, and the range comprises the outputs.

Sometimes the domain of a function, the allowable inputs, is specifically stated. When a function is given by a formula but the domain is not stated, the domain is assumed to be the set of all inputs for which the formula makes sense. That is, the domain is the set of all inputs for which the formula gives a real number as the output.

For example, 3 is in the domain of $f(x)=\dfrac{2+x}{x}$ because if we put 3 into the formula we get a perfectly reasonable answer:

$$f(3)=\frac{2+3}{3}=\frac{5}{3}$$

On the other hand, 0 is not in the domain of f because insertion of 0 into the formula results in division by 0, which is never allowed. In fact, we get a meaningful output as long as we do not use the input $x=0$. Hence the domain of f is

$$(-\infty, 0)\cup(0, \infty)$$

STEP-BY-STEP STRATEGY: Finding the Domain of a Function

When a function is given by a formula and no domain is specified, the domain is understood to be all real numbers for which the formula makes sense. That is, the domain is the set of all inputs for which the formula gives a real number as the output.

Step 1 Exclude all real numbers for which the formula does not make sense (for example, numbers that would result in division by 0, square roots of negative numbers, or 0 to the 0 power).

Step 2 The domain consists of all real numbers not excluded in the previous step.

EXAMPLE 1.4 Finding the Domain

In each part, find the domain of the given function.

a. $g(t)=3t^2+2$

b. $f(x)=\dfrac{\sqrt{x-3}}{x-7}$

c. $k(t)=\dfrac{t^2}{9t}$

SOLUTION

a. The formula $g(t)=3t^2+2$ makes sense for all values of t. Thus, the domain of g is the set $\mathbb{R}$ of all real numbers.

b. We look for values of x for which the formula does not make sense and exclude them. We note first the expression $x-7$ in the denominator. The denominator of a fraction cannot be 0, so we must exclude $x=7$ from the domain. Now look at the numerator. We cannot take the square root of a negative number, so we must have $x-3\geq 0$ or $x\geq 3$. We conclude that the formula makes sense for all real numbers greater than or equal to 3 but not equal to 7. Using interval notation, we write the domain as

$$[3, 7)\cup(7, \infty)$$

c. The formula does not make sense when the denominator is 0, so we exclude $t=0$ from the domain. There are no other restrictions, so the domain is

$$(-\infty, 0)\cup(0, \infty)$$

TRY IT YOURSELF 1.4 Brief answers provided at the end of the section.

Find the domain of the function $h(x) = \sqrt{x-1}$.

EXTEND YOUR REACH

a. Here is a "solution" of part c of the example that contains a significant error. Cancel the common t from both numerator and denominator to obtain $\frac{t^2}{9t} = \frac{t}{9}$. Because this last expression makes sense for all real numbers, we conclude that the domain of $k(t)$ is the set of all real numbers. Explain what is wrong with this "solution."

b. Bearing in mind that the expression $\frac{t^2}{9t}$ is not defined when $t = 0$, sketch the graph of the equation $y = \frac{t^2}{9t}$.

c. Does the graph you made in part b suggest a reasonable way to extend the definition of $k(t)$ so that its domain is the set of all real numbers?

The **range** of a function is the set of all function outputs.

Thus far we have focused on the inputs for a function. Now we focus on the function outputs, which constitute the **range**. These are the values taken by the function as the input varies through the domain.

For example, 8 is in the range of $g(x) = x^2 + 7$ because $g(1) = 8$. On the other hand, 5 is not in the range of g: There is no x so that $x^2 + 7 = 5$ (because $x^2 \geq 0$ for all real x).[1]

EXAMPLE 1.5 Finding the Domain and Range

In each part, find the domain and range of the given function.

a. $g(s) = s^2 + 5$

b. The constant function $h(x) = 3$

SOLUTION

a. The formula makes sense for all choices of s, so the domain is $\mathbb{R}$. The range is the set of all function outputs. Because $s^2 \geq 0$, the formula $g(s) = s^2 + 5$ gives real numbers that are 5 or larger. In fact, every real number that is 5 or larger is a value of the function. We conclude that the range of g is $[5, \infty)$.

b. Note that h is a function whose value $h(x)$ is the number 3, regardless of the value of x. So $h(0) = 3$, $h(3) = 3$, and $h(x^2 + 1) = 3$ for every x. Because the formula makes sense for every value of x, the domain is $\mathbb{R}$. The only output is the single number 3, so the range is $\{3\}$.

TRY IT YOURSELF 1.5 Brief answers provided at the end of the section.

Find the range of the function $h(x) = \sqrt{x-1}$.

[1]The equation does have a solution if we allow complex numbers, but in general, we will restrict our attention to real numbers unless we state otherwise.

Equations and Functions

An equation may determine a function.

Some equations define functions, and some do not. For example, the equation $x + y^2 - 2y = 0$ does not determine y as a function of x. To understand this, consider $x = 0$. Observe that both $y = 0$ and $y = 2$ satisfy the equation when $x = 0$. So the formula assigns to $x = 0$ more than one y-value. On the other hand, the equation $x + y = 7$ does define y as a function of x, namely the function $y = 7 - x$.

STEP-BY-STEP STRATEGY: Deciding Whether an Equation Determines a Function

To determine whether an equation in x and y determines y as a function of x, proceed as follows:

Step 1 Solve the equation for y.

Step 2 The equation determines y as a function of x provided the equation has at most one solution for each value of x.

EXAMPLE 1.6 Deciding Whether an Equation Defines a Function

Decide whether the following equations define y as a function of x. Justify your answer.

a. $x^2 + y^2 = 1$

b. $y + x^2 = 3 - y$

SOLUTION

a. We solve the equation for y:

$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2 \qquad \text{Move } x^2 \text{ right.}$$

$$y = \pm\sqrt{1 - x^2} \qquad \text{Take square roots.}$$

We can see here that for some values of x there are two solutions; hence, some inputs have more than one output.

An alternative approach is to let $x = 0$ in the equation $x^2 + y^2 = 1$. We get $y^2 = 1$, which has two solutions, $y = 1$ and $y = -1$. Because this x-value does not result in a unique y-value, the equation does not define y as a function of x.

b. We can solve the equation for y:

$$y + x^2 = 3 - y$$

$$2y = 3 - x^2 \qquad \text{Add } y \text{ and subtract } x^2.$$

$$y = \frac{3 - x^2}{2} \qquad \text{Divide by 2.}$$

This equation *does* define y as a function of x because it assigns to each value of x exactly one value of y, namely $y = \dfrac{3 - x^2}{2}$.

TRY IT YOURSELF 1.6 Brief answers provided at the end of the section.

Does the equation $(3y - x)(2y - x) = 0$ define y as a function of x? Justify your answer.

EXTEND YOUR REACH Does the equation $x + |y| = 1$ define y as a function of x? Does it define x as a function of y?

Graphs and Functions

Graphs are visual representations of functions.

The **graph of a function** f is the graph of the equation $y = f(x)$.

The **graph of a function** is the most common and important example of a graph of an equation. For example, the graph of $f(x) = x^3$ is just the graph of the equation $y = x^3$ shown in Figure 1.3.

Some graphs of equations define functions, but some do not. For example, the graph in Figure 1.4 defines y as a function of x. Note the dashed vertical line in the figure. A graph defines a function precisely when each vertical line hits the graph no more than once. This is because the number of times the vertical line through x crosses the graph determines the number of y-values assigned to x. This is a useful visual test for deciding whether a graph represents a function.

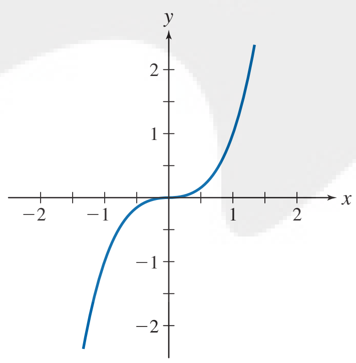


Figure 1.3 The graph of the function $f(x) = x^3$

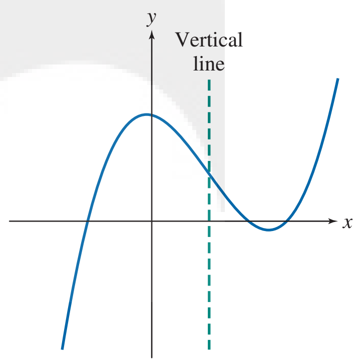


Figure 1.4 The vertical line test: This graph defines a function because each vertical line intersects the graph exactly once (if at all).

LAWS OF MATHEMATICS: The Vertical Line Test

A graph represents a function provided that each vertical line intersects the graph no more than once. If there is *any* vertical line that intersects the graph more than once, the graph does not represent a function.

EXAMPLE 1.7 Graphs and Functions

Determine whether the graph in **Figure 1.5** defines y as a function of x.

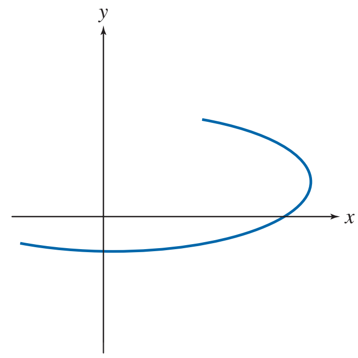


Figure 1.5 Graph for Example 1.7

SOLUTION

The dashed vertical line shown in **Figure 1.6** intersects the graph twice, so the graph fails the vertical line test. It does not define y as a function of x. It is worth noting that some vertical lines do intersect the graph in Figure 1.5 exactly once, but this is not sufficient for a function to be defined. For a graph to pass the vertical line test, *every* vertical line must intersect the graph at most once.

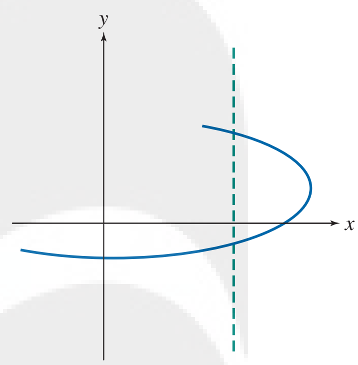
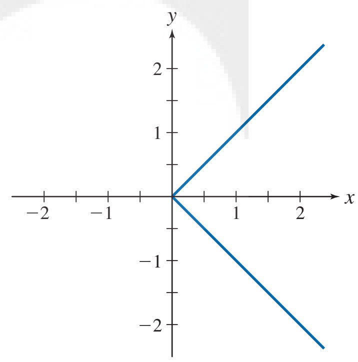


Figure 1.6 The vertical line test: This graph fails the vertical line test.

TRY IT YOURSELF 1.7 Brief answers provided at the end of the section.

Does the graph in **Figure 1.7** define y as a function of x?



DF **Figure 1.7** Graph for Try It Yourself 1.7

EXTEND YOUR REACH

a. Does the graph in **Figure 1.8** define y as a function of x? Note that some vertical lines will not meet the graph at all.

b. Start with the graph of $y = x$. For every rational number x, replace the point (x, x) on the graph by $(x, -x)$. Does this graph pass the vertical line test? Can you think of a way to draw this graph?

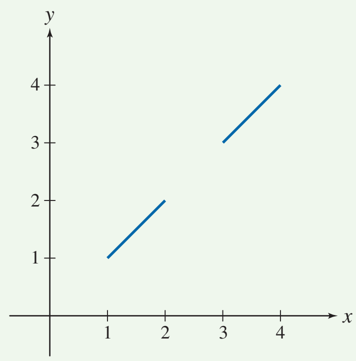


Figure 1.8 Applying the vertical line test

CONCEPTS TO REMEMBER: The Basics of Functions

1. A function consists of a domain and a rule that assigns to each element of the domain *exactly* one element of some collection.
2. A function $y = f(x)$ may be defined by a formula, words, a graph, or a table of values.
3. For a function $f(x)$ given by a formula, the domain is assumed to be the set of all x-values for which the formula makes sense, unless specifically stated otherwise.
4. The range of a function is the set of all values taken by the function. For a function given by a formula $y = f(x)$, the range is the set of all values of c for which the equation $f(x) = c$ has a solution x in the domain of f.
5. A graph in terms of x and y determines y as a function of x precisely when each vertical line does not intersect the graph more than once.

MODELS AND APPLICATIONS Representing Costs of Everyday Purchases

When functions are used in the real world, it is important to use functional notation properly and think carefully about the domain and range.

EXAMPLE 1.8 Cost of Sodas

If $C(s)$ is the cost, in dollars, of purchasing s sodas, use functional notation to indicate the cost of purchasing a six-pack of sodas.

SOLUTION

In functional notation, the cost, in dollars, of purchasing six sodas is $C(6)$. It is important to emphasize that the expression $C(6)$ has a perfectly valid meaning, even if we cannot calculate its value.

TRY IT YOURSELF 1.8 Brief answers provided at the end of the section.

If $P(f)$ is the number of pints of paint needed to paint f square feet, explain the meaning of $P(21)$ in terms of pints of paint.

When a function is given by a formula, it is understood that the domain and range are determined strictly by the formula. But in applications, additional constraints may apply.

EXAMPLE 1.9 Domain and Range in an Application

A child sells lemonade in a neighborhood of 200 people. Her total income I depends on the number n of glasses of lemonade sold. The relationship is $I = 0.50n$ dollars. Assuming that the maximum possible number of glasses sold is the population of the neighborhood, determine the domain and range of I.

SOLUTION

She can sell at most 200 glasses of lemonade. Thus, the domain is $[0, 200]$. Her total income varies from 0 dollars to $0.50 \times 200 = 100$ dollars. That gives a range, in dollars, of $[0, 100]$.

TRY IT YOURSELF 1.9 Brief answers provided at the end of the section.

What are the domain and range if she is guaranteed the sale of a glass to each of her parents and both of her brothers?

TRY IT YOURSELF ANSWERS

1.1 **a.** $g(9) = 39$
b. $h(t-1) = (t-1)^2 - 1 = t^2 - 2t$

1.2 $V(60) = 4.0$ volts. The voltage measured after 60 minutes is 4.0 volts.

1.3 $f(-1) = -2$

1.4 The domain is $[1, \infty)$.

1.5 The range is $[0, \infty)$.

1.6 The equation does not define y as a function of x because it assigns to each value of x both $y = \dfrac{x}{2}$ and $y = \dfrac{x}{3}$.

1.7 The graph does not define y as a function of x because it fails the vertical line test: Some vertical lines intersect the graph more than once.

1.8 The expression $P(21)$ is the number of pints of paint needed to paint 21 square feet.

1.9 Domain $[4, 200]$. Range $[2, 100]$.

EXERCISE SET 1.1

CHECK YOUR UNDERSTANDING

1. How many outputs can a function assign to a given input that is in the domain?
2. **True or false:** Functional notation such as $f(7)$ makes sense only if we can calculate its value.
3. What test is used to determine whether a graph is the graph of a function?
4. **True or false:** All functions are defined by formulas.
5. To say that t is in the domain of a function f means ____________.
6. The graph of a function f is the graph of an equation. Which equation?
7. **True or false:** If a graph passes the vertical line test, then it is the graph of a function.
8. A certain equation has the same solution for y if $x = 4$ or $x = 9$. Does this mean the equation does not define y as a function of x?

SKILL BUILDING

Finding the domain. In Exercises 9 through 18, state the domain of the given function.

9. $f(x) = \dfrac{x}{x-1}$

10. $f(x) = \dfrac{1}{\sqrt{x}}$

11. $f(x) = x^2 + \dfrac{x}{3}$

12. $f(x) = \dfrac{1}{x} + \dfrac{1}{x+1}$

13. $h(x) = \dfrac{1}{x} + \dfrac{1}{x+1}$, $x \geq -\dfrac{1}{2}$

14. $f(x) = \sqrt{x-7}$

15. $f(x) = 2^{1-x} + 3^{x-1}$

16. $y = \dfrac{x}{x^2 - 3x + 2}$

17. $y = \dfrac{x^2}{x}$

18. $y = \dfrac{1}{x^3 - 3x^2 + 2x}$

Calculating function values. In Exercises 19 through 32, calculate the indicated function value. Simplify your answer.

19. $f(x) = \dfrac{x}{x+1}$. Calculate $f(2)$.

20. $f(x) = x^2 + x^{-2}$. Calculate $f(-2)$.

21. $g(t) = \sqrt{t^2 - 5}$. Find $g(3)$.

22. $h(x) = (x-1)^2$. Find $h(x+1)$.

23. $k(\omega) = \sqrt{\omega}$. Find $k(\omega^2)$.

24. $x(t) = \dfrac{t}{t+1}$. Find $x\left(\dfrac{1}{t-1}\right)$, $t \neq 0$.

25. $h(y) = \dfrac{1}{\sqrt{y}}$. Find $h\left(\dfrac{1}{z^2}\right)$, $z \neq 0$.

26. $\sigma(s) = (s-2)(s-4) + 1$. Find $\sigma(s+3)$.

27. $f(x) = x - \dfrac{1}{x}$. Calculate $f\left(\dfrac{1}{x}\right)$, $x \neq 0$.

28. $f(x) = x^2$. Calculate $f(x^2)$.

29. $f(x) = \sqrt{x+1}$. Calculate $f(x^2 + 2x)$, assuming that x is positive.

30. $f(x) = x^2 - h^2$. Calculate $f(x+h)$.

31. $f(x) = \dfrac{x}{x+1}$. Calculate $f(f(x))$, $x \neq -1$.

32. $f(x) = \dfrac{x+1}{x-2}$. Calculate $f\left(\dfrac{2x+1}{x-1}\right)$, $x \neq 1$.

Difference quotients. Exercises 33 through 39 involve difference quotients, which are fractions involving function values. They arise in the calculation of derivatives in calculus. Calculate the quotient and simplify.

33. $f(x) = 2x + 1$. Simplify $\dfrac{f(x) - f(a)}{x - a}$, $x \neq a$.

34. $f(x) = 2x + 1$. Simplify $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$.

35. $f(x) = x^2$. Simplify $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$.

36. $f(x) = x^2$. Simplify $\dfrac{f(x) - f(a)}{x - a}$, $x \neq a$.

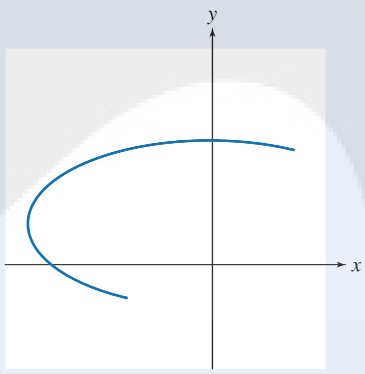
37. $f(x) = \sqrt{x}$. Show that $\dfrac{f(x) - f(a)}{x - a} = \dfrac{1}{f(x) + f(a)}$ if $x \neq a$ and both x and a are positive.

38. $f(x) = \sqrt{x}$. Show that $\dfrac{f(x+h) - f(x)}{h} = \dfrac{1}{\sqrt{x+h} + \sqrt{h}}$ if both x and h are positive.

39. $f(x) = \dfrac{1}{x}$. Simplify $\dfrac{f(x) - f(y)}{x - y}$ if $x \neq 0$, $y \neq 0$, and $x \neq y$.

Graphs and functions. In Exercises 40 through 45, determine whether the given graph is the graph of a function.

40.



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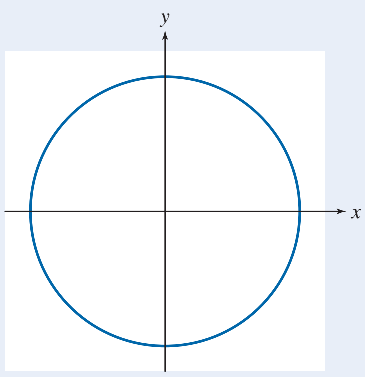

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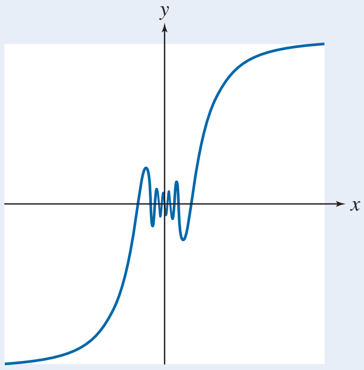

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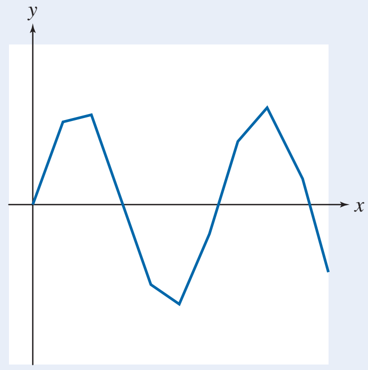

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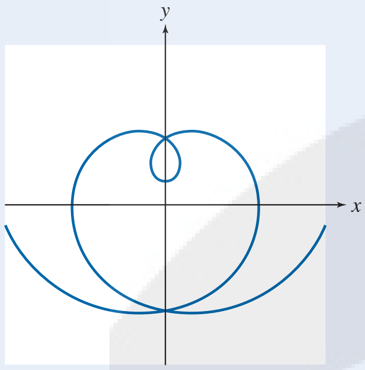

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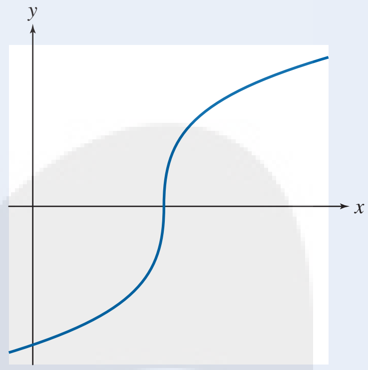


PROBLEMS

Range of a function. In Exercises 46 through 54, find the range of the given function.

46. $f(x) = 7 + 3x$

47. $f(x) = \sqrt{x}$

48. $f(x) = \sqrt{x+1}$

49. $f(x) = 1 + \dfrac{1}{x}$

50. $f(x) = x^3 + 1$

51. $f(x) = \dfrac{x^2}{x}$

52. $f(x) = \dfrac{x+1}{x}$

53. $f(x) = \sqrt{x^2 + 1}$

54. $f(x) = (x+1)^2$

Piecewise-defined functions.

55. Define $g(x) = \begin{cases} x^3 + 1 & \text{if } x \le 2 \\ x^2 & \text{if } x > 2. \end{cases}$

Calculate $g(7)$, $g(1)$, and $g(2)$.

56. Bill says that g defined by

$$g(x) = \begin{cases} x + 3 & \text{if } x \le 0 \\ x + 1 & \text{if } x \ge 0 \end{cases}$$

is a piecewise-defined function, but Alice says it is not. Who is right, and why?

57. Define $f(t) = \begin{cases} \dfrac{t^2 - 1}{t - 1} & \text{if } t \ne 1 \\ 2 & \text{if } t = 1. \end{cases}$

Find $f(1)$ and $f(2)$.

58. Define $h(x) = \begin{cases} x - 1 & \text{if } x < 2 \\ 9 & \text{if } x = 2 \\ x + 1 & \text{if } x > 2. \end{cases}$

Find $h(1)$, $h(2)$, and $h(3)$.

59. Define $f(x) = \begin{cases} x + 1 & \text{if } x \le 5 \\ x + 6 & \text{if } x > 5. \end{cases}$

Calculate $f(x+1)$.

The greatest integer function. The greatest integer function $[x]$ is the integer n such that $n \le x < n+1$. It gives the largest integer less than or equal to x. For example, $[3.1] = 3$ and $[-3.1] = -4$.

60. Calculate $[5.8]$.

61. Calculate $[\sqrt{72}]$.

62. Calculate $[-\sqrt{44}]$.

63. Calculate $[29]$.

64. Is it true that $|[x]| = [|x|]$? Justify your answer.

65. Is it true that $[[x]] = [x]$? Justify your answer.

Functions given by equations. In Exercises 66 through 72, determine whether the given equation determines y as a function of x. Explain your answer.

66. $2x^2y + 5y = 2x^2$

67. $\sqrt{xy} = 1,\ x \neq 0$

68. $x - y = xy,\ x \neq -1$

69. $y^2 - 2x^2 = 1$

70. $x + y^2 = 4$

71. $xy + 1 = x,\ x \neq 0$

72. $x^y = 1,\ x \neq 0$ and $x \neq 1$

Is it a function? In Exercises 73 through 79, determine whether the given assignment determines a function. If it does not, explain why.

73. The domain is all living people. Assign to each person the names of his or her uncles.

74. The domain is the decades from 1790 through 2020. Assign to each decade the average U.S. population over the decade.

75. The domain is the collection of all mothers. Assign to each mother her children.

76. The domain is the collection of all living children. Assign to each child its birth mother.

77. The domain is the set of all real numbers. The range is the set of all real numbers. Assign to each number all numbers no farther away than a distance of 1 from the given number.

78. The domain consists of all allowable weights, in ounces, of first-class domestic letters. The range is a collection of prices. The assignment $p(x)$ is the current postage required to mail a first-class domestic letter that weighs x ounces.

79. The domain is the collection of all U.S. presidents. The range is all years from 1790 to the present. Assign to each president the years served.

Graphing functions. In Exercises 80 through 84, graph the given function. You may want to consult a table of values.

80. $f(x) = 1 - x^2$

81. $f(x) = \sqrt{x}$

82. $f(x) = \dfrac{1}{x}$

83. $f(x) = x + \sqrt{x}$

84. $f(x) = x - x^2$

Making graphs. In Exercises 85 through 90, use a graphing utility to graph the indicated function. The suggested horizontal span is an indication of the x-values to include.

85. $g(x) = \sqrt{x^3 - x}$. We suggest a horizontal span from $x = -1$ to $x = 2$.

86. $h(x) = \dfrac{x}{x^2 + 1}$. We suggest a horizontal span from $x = -3$ to $x = 3$.

87. $f(t) = t^3 2^{-t}$. We suggest a horizontal span from $t = -1$ to $t = 10$.

88. $f(x) = \left(1 + \dfrac{1}{x}\right)^x$. We suggest a horizontal span from $x = 0$ to $x = 10$.

89. $f(x) = \dfrac{x}{x^2 - 1}$. We suggest a horizontal span from $x = -5$ to $x = 5$.

90. **The π function.** The function $\pi(x)$ gives the number of primes less than or equal to x. Plot the graph of $\pi(x)$ on a span of 0 to 100. (Note that you will need a computer algebra system such as Maple, Mathematica, or MATLAB to make this graph.) When would we expect $\pi(x)$ to be different from $\pi(x+1)$?

Using a graph to find range. In Exercises 91 through 94, use a graphing utility to plot the graph of the given function. Use the graph to find the range of the function. Pay attention to the given restrictions on the domain.

91. $f(x) = \dfrac{2x}{x^2 + 1}, \quad -5 \le x \le 5$

92. $f(x) = 3^x - 2^x, \quad 0 \le x \le 2$

93. $f(x) = x(1 + 2^{-x}), \quad 0 \le x \le 2$

94. $\dfrac{7 + 2x^2}{11 + x^2}, \quad -2 \le x \le 2$

MODELS AND APPLICATIONS

Determining domain and range from practical constraints.

95. **A rock falling.** If we drop a rock from 256 feet above the surface of Earth and ignore air resistance, after t seconds the rock will fall $D(t) = 16t^2$ feet. This formula is valid until the rock strikes the ground.

   a. The rock strikes the ground when $D(t) =$ 256 feet. What is the value of t when the rock strikes the ground?

b. What is the domain of this function?

c. What is the range of this function?

96. Projectile path. A projectile is fired from the origin and follows the path of the parabola $y = \dfrac{x(400-x)}{400}$ until it strikes the ground. (Both x and y are measured in feet.) The graph of $y = \dfrac{x(400-x)}{400}$ is shown in Figure 1.9. Let $H(x)$ denote the height, in feet, of the projectile x feet downrange. Find the domain and range of H.

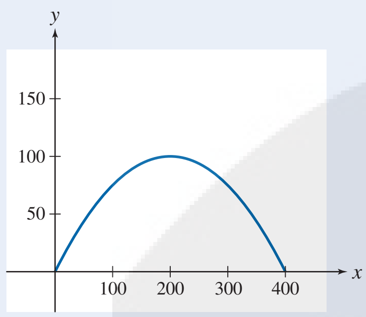


Figure 1.9 Graph of $y = x(400-x)/400$

97. A weekly newspaper. A weekly newspaper is sold locally for \$1.25 per copy in a town with a population of 13,000. Let $R(n)$ be the amount, in dollars, received from the sale of n newspapers this week. Practically speaking, what is the domain of R? (Strictly speaking, n takes values in the integers, but you may assume that n can take other values.) What is the range of R in the best possible case for the paper?

98. An investment. A man invests \$5000 in an account that draws interest. He leaves the money in the account for 25 years and then closes the account. The balance B of the account after t years is given by

$$B(t) = 5000 \times 1.06^t \text{ dollars}$$

Find the domain and range of B.

99. Another investment. A woman invests in an account that draws interest. Let $B(t)$ denote the balance, in dollars, of the account after t years. Explain in terms of the balance of the account what $B(5)$ means.

100. Cooking a yam. A yam is placed in the oven to bake. Let $Y(t)$ denote the temperature, in degrees Fahrenheit, of the yam after t minutes in the oven. Explain what $Y(3)$ means in terms of the temperature of the yam.

101. A roof line. The automobile shop shown in Figure 1.10 is 50 feet wide. If we take $t = 0$ to be the location of the west wall, then the height of the roof at a distance of t horizontal feet from the west wall is given by $H = 11 + 0.5t$ feet. This formula is valid until we reach the peak, 25 feet from the west wall. If instead we think of $t = 0$ as the location of the peak, then the roof from the peak to east wall follows $H = 23.5 - 0.5t$ feet, where t is the horizontal distance in feet from the peak. This is valid up to the east wall 25 feet from the peak. Using $T = 0$ as the horizontal distance in feet from the west wall, find a piecewise-defined function that gives the height of the roof in terms of T.

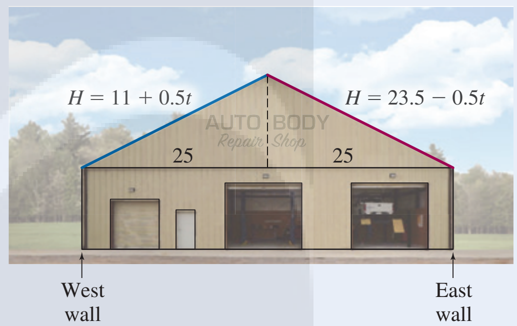


Figure 1.10 An automobile shop

102. A rising river. The measurement of snowfall in the mountains allows city planners to model the projected depth of a river running through the city over the next year. The model is

$$D = \frac{10(t-175)^2 + 75{,}000}{2500 + (t-175)^2}$$

where D is the depth in feet and t is the time in days since January 1.

a. Plot the graph of river depth over a 1-year period.

b. Flood danger occurs when the river reaches a depth of 33 feet. Does the city need to prepare for a flood this year?

CHALLENGE EXERCISES FOR INDIVIDUALS OR GROUPS

103. A leaky can. Suppose that a cylindrical can is 10 inches tall and full of tea. It springs a leak in the side, and the tea begins to stream out. The depth H, in inches, of the tea remaining in the can is a function of the distance D in inches (measured from the base of the can) at which the stream of tea strikes the ground. This situation is shown in Figure 1.11. The relationship is given by $H(D) = \frac{D^2}{4} + 1$. This formula is valid only until the height of the tea has decreased to the height of the leak.

a. Do larger values of D correspond to larger or smaller values of H?

b. What is the value of D when $H = 10$?

c. What is the domain of H? *Suggestion:* Your answer from part b should be helpful.

d. At what height H above the bottom of the can is the leak? *Suggestion*: Observe that, when the height of the tea reaches the height of the leak, the tea will stop flowing. Thus, $D = 0$ when the height of the tea reaches the height of the leak.

e. What is range of H?

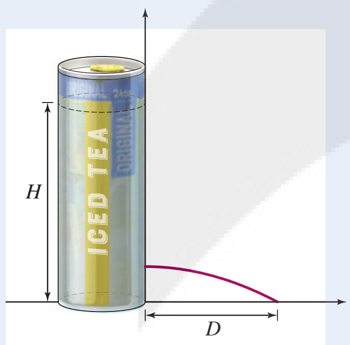


Figure 1.11 A leaking can of tea

Periodic functions. Exercises 104 through 111 are concerned with periodic functions. A function f is said to be periodic if there is a number p such that $f(x+p) = f(x)$ for all x in the domain of f. The period of such a function is the smallest positive value of p that makes the preceding formula valid (assuming such a value exists). Intuitively, a periodic function is one that repeats in a predictable fashion. The period is the smallest time that it takes to repeat. For example, the display on a digital clock representing minutes starts at 0, increases to 59, and then starts over at 0. If $M(t)$ is the minute part of the time shown by a digital clock after t minutes starting at 0, then $M(t+60) = M(t)$, and M is periodic with period 60 minutes.

104. How long until my birthday? For this problem, ignore leap years and assume that each year is exactly 365 days long. For a given day of the year, let B be the number of days left until your next birthday. Is B periodic? If so, what is its period?

105. Around a circle. A point moves around a circle at a constant speed of one trip around the circle each hour. Let $L(t)$ denote the location of the point t minutes after the motion begins. Is L periodic? If so, what is the period of L?

106. Around the circle again. A point starts at a particular place on a circle of radius 1 foot and moves in a counterclockwise direction around the circle. Let $P(d)$ denote the position of the point when it has traveled d feet. Is P periodic? If so, what is its period?

107. Around the circle yet again. A point begins at a particular location on a circle. It moves counterclockwise around the circle. Its speed increases by 1 unit each time it completes a revolution of the circle. Let $L(t)$ be the location of the point t minutes after motion began. Explain why L is not periodic.

108. More repetition. Assume that f is a periodic function with period p. How does $f(x)$ compare with $f(x+2p)$? How about $f(x+np)$ for a positive integer n?

109. Periodic or not. Suppose f is periodic with period p. Which of the following functions are periodic? If they are periodic, what is the period?

a. $g(x) = f(x)+1$

b. $g(x) = -f(x)$

c. $g(x) = 2f(x)$

d. $g(x) = f(2x)$

e. $g(x) = f(x-1)$

110. Division. For a positive integer n, let $R(n)$ denote the remainder when n is divided by 17. Is R periodic? If so, what is its period?

111. A variant of the greatest integer function. The greatest integer function $[x]$ is the integer n such that $n \le x < n+1$. Is the function $f(x) = x - [x]$ periodic? If so, what is its period?

REVIEW AND REFRESH: Exercises from Previous Sections

112. From Section P.1: Find the equation of the line that passes through $(3, -2)$ and has slope 3.

Answer: $y = 3x - 11$

113. From Section P.1: Find the center and radius of the circle with the equation $x^2 + y^2 = 6 + 4x + 2y$.

Answer: Center $(2, 1)$ and radius $\sqrt{11}$

114. From Section P.2: Solve the inequality $3x - 1 > x + 5$.

Answer: $(3, \infty)$

115. From Section P.2: Solve the inequality $|x - 8| < 5$.

Answer: $(3, 13)$

116. From Section P.3: Solve the inequality $x^2 + 4x - 12 > 0$.

Answer: $(-\infty, -6) \cup (2, \infty)$

117. From Section P.3: Solve the inequality $x^3 < x^2$.

Answer: $(-\infty, 0) \cup (0, 1)$

1.2 Average Rates of Change

1.1 The Basics of Functions
1.2 Average Rates of Change
1.3 Graphs and Rates of Change
1.4 Limits and End Behavior of Graphs

The average rate of change is one way to measure change over an interval.

In this section, you will learn to:

1. Calculate the average rate of change of a function on an interval.
2. Interpret the average rate of change in applied settings.
3. Use the average rate of change to estimate function values in applied settings.

If a child grew 10 inches over a period of 10 years, we would immediately conclude that the child had grown by an average of 1 inch per year. Of course, that does not mean that growth was exactly 1 inch each year—that just isn't how people grow. But it is a reasonable way of describing the growth of the child based on limited information. A growth of 1 inch per year is an average rate of change, and it is an entirely intuitive notion.

It is also useful. How much growth would be expected next year? Well, 1 inch is a reasonable guess, and the best guess you can make, given the information you have. You don't expect your guess to be exactly right, and indeed, you know that the child will stop growing someday.

The average rate of change for any function works just like the average rate of change for the growing child. Calculating rates of change for functions is fundamental in both mathematics and its applications to real-world problems.

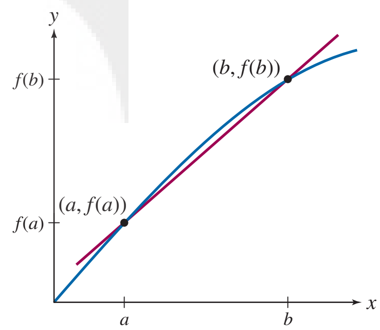


Figure 1.12 The secant line

Secant Lines

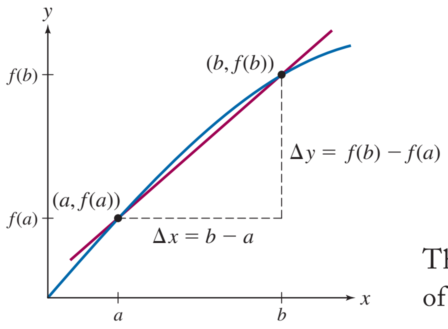
A secant line can be used to measure the rate of change of a function.

We can associate a **secant line** to every pair of points on the graph of a function. **Figure 1.12** shows the secant line in relation to the graph.

The **average rate of change** of a function on an interval is the slope of the associated secant line.

The **secant line** for a function f for the interval $[a, b]$ is the line through the points $(a, f(a))$ and $(b, f(b))$.

The **average rate of change** of a function $y = f(x)$ on the interval $[a, b]$ is the slope of the secant line for $[a, b]$.



DF **Figure 1.13** The average rate of change

The formula for the average rate of change is

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

This is also called the average rate of change from $x = a$ to $x = b$. The average rate of change is the change in y over the change in x. **Figure 1.13** illustrates the numerator and denominator in this definition. The expression $\dfrac{f(b) - f(a)}{b - a}$ is called a difference quotient.

EXAMPLE 1.10 Calculating the Average Rate of Change

Calculate the average rate of change of $f(x) = x^2$ on the interval $[2, 4]$.

SOLUTION

Because $f(2) = 2^2 = 4$ and $f(4) = 4^2 = 16$, we have

$$\begin{aligned}\text{Average rate of change} &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{16 - 4}{4 - 2} \\ &= 6\end{aligned}$$

This average rate of change is the slope of the secant line for the interval $[2, 4]$. The graph of f along with the secant line is shown in **Figure 1.14**.

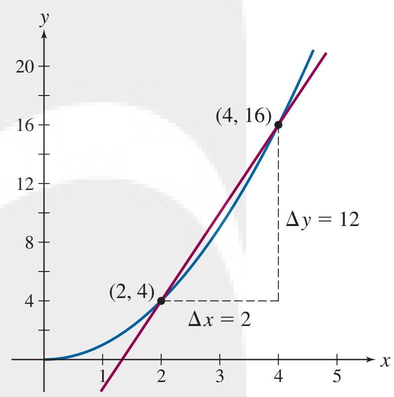


Figure 1.14 A secant line: The slope of the secant line is the average rate of change.

TRY IT YOURSELF 1.10 Brief answers provided at the end of the section.

Calculate the average rate of change of $f(x) = \dfrac{1}{x}$ from $x = 1$ to $x = 3$.

The next example shows how the average rate of change depends on the length of the interval. This is a key step in calculating the derivative in calculus.

EXAMPLE 1.11 More on Calculating Average Rates of Change

Let $f(x) = \sqrt{x}$. Show that if both a and h are positive, then the average rate of change of f on the interval $[a, a + h]$ is $\dfrac{1}{\sqrt{a + h} + \sqrt{a}}$.

SOLUTION

We calculate the average rate of change as follows:

$$\begin{aligned}\text{Average rate of change} &= \frac{f(a+h)-f(a)}{(a+h)-a}\\ &= \frac{\sqrt{a+h}-\sqrt{a}}{(a+h)-a}\\ &= \frac{\sqrt{a+h}-\sqrt{a}}{h}\end{aligned}$$

To change this expression to the required form, we rationalize the numerator of our fraction by multiplying the numerator and denominator by $\sqrt{a+h}+\sqrt{a}$, the conjugate of the numerator. Then we use the fact (based on factoring the difference of squares) that

$$(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B}) = A-B$$

We find

$$\begin{aligned}\frac{\sqrt{a+h}-\sqrt{a}}{h} &= \frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})} && \blacktriangleleft \text{ Multiply top and bottom by conjugate.}\\ &= \frac{(a+h)-a}{h(\sqrt{a+h}+\sqrt{a})} && \blacktriangleleft \text{ Use } (\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B}) = A-B.\\ &= \frac{\cancel{h}}{\cancel{h}(\sqrt{a+h}+\sqrt{a})} && \blacktriangleleft \text{ Cancel common factors.}\\ &= \frac{1}{\sqrt{a+h}+\sqrt{a}} && \blacktriangleleft \text{ Simplify.}\end{aligned}$$

TRY IT YOURSELF 1.11 Brief answers provided at the end of the section.

Show that if both a and b are nonzero, the average rate of change of $g(x) = \frac{1}{x^2}$ on the interval $[a, b]$ is $-\frac{a+b}{a^2b^2}$.

EXTEND YOUR REACH

a. For the function $y = x^2$, calculate the average rate of change on each of the following intervals. In each case, give your answer correct to three decimal places.

- From $x = 1$ to $x = 2$
- From $x = 1$ to $x = 1.5$
- From $x = 1$ to $x = 1.25$
- From $x = 1$ to $x = 1.1$

b. As the interval gets shorter, does it appear that the average rate of change is approaching a specific number? If so, identify that number.

MODELS AND APPLICATIONS Ants, Ebola, and Temperature

The idea of an average rate of change is common in daily life. For example, if you drive 30 miles in half an hour, your average speed is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{30}{\frac{1}{2}} = 60 \text{ miles per hour}$$

In this context, the average rate of change is just your average speed. There are always units associated with the average rate of change: the units of the function divided by the units of the independent variable. In this case, the units are miles per hour. Units are always important, and getting the units right can often facilitate the calculation and use of the average rate of change.

EXAMPLE 1.12 Units and Interpreting Average Rates of Change: An Ant Colony

An ant colony is monitored in a laboratory. It is found that the population N of the colony after t weeks is given by

$$N(t) = 200 \times 1.2^t \text{ ants}$$

a. Calculate the average rate of change of the population from week 3 to week 5. Round your answer to the nearest whole number, and be sure to include proper units.

b. Explain the meaning of the calculation you made in part a in terms of the ant colony.

SOLUTION

a. We note that formally the units for the average rate of change are the units of the function divided by the units of the independent variable, so in this case, the units are ants per week. But the formality is really not needed because the average rate of change is the change in population divided by the number of weeks—that is, ants per week.

We want to find the average rate of change of N on the interval $[3, 5]$:

$$\begin{aligned}\text{Average rate of change} &= \frac{N(5) - N(3)}{5 - 3} \\ &= \frac{(200 \times 1.2^5) - (200 \times 1.2^3)}{2} \\ &\approx 76 \text{ ants per week}\end{aligned}$$

b. The average rate of change of 76 ants per week is the average increase in the ant population per week over the 2-week period. In other words, during this period, the population grew by an average of 76 ants each week. The graph of the ant population along with the secant line we used to calculate the average rate of change is shown in Figure 1.15.

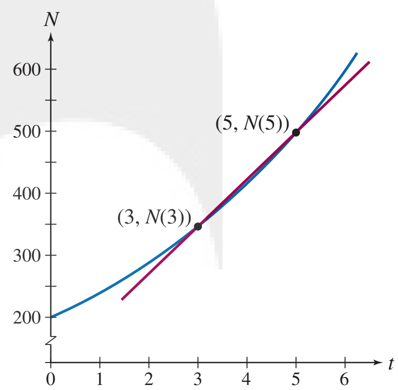


Figure 1.15 Graph of ant population and secant line

TRY IT YOURSELF 1.12 Brief answers provided at the end of the section.

Find the average rate of change per week of the ant population from week 3 to week 7. Round your answer to the nearest whole number, and be sure to include proper units.

EXAMPLE 1.13 Using the Average Rate of Change from a Table: Ebola

The accompanying table shows the cumulative number of cases in West Africa of Ebola, a deadly virus, reported by the World Health Organization for the given dates in 2015.

Day	October 1	October 16	October 22	October 26
Cumulative cases	28,408	28,468	28,504	28,528

Let $N(d)$ denote the cumulative number of Ebola cases reported d days after October 1. The plot of N is shown in **Figure 1.16**. Note that the vertical axis begins at 28,000.

a. Calculate the average rate of change in the cumulative number of cases from October 1 to October 16. Be sure to give proper units.

b. Explain the meaning of the number you calculated in part a in terms of the number of new cases of Ebola.

c. Use the average rate of change to estimate the cumulative number of cases reported by October 9.

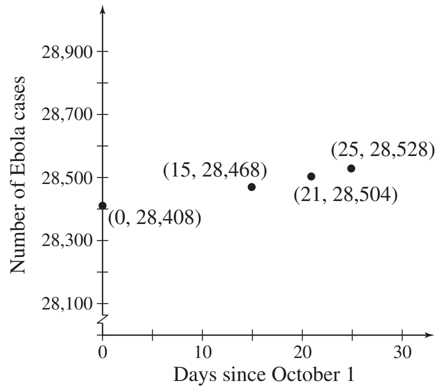


Figure 1.16 Plot of Ebola data

SOLUTION

a. October 1 corresponds to $d = 0$, and October 16 corresponds to $d = 15$. The change in the cumulative number of cases N over this time period is

$$28,468 - 28,408 = 60 \text{ new cases reported}$$

This change occurred over a 15-day period. Therefore,

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in } N}{\text{Change in } d}\\ &= \frac{60}{15}\\ &= 4 \text{ new cases per day}\end{aligned}$$

b. During the period from October 1 to October 16, there were, on average, four new cases of Ebola reported each day.

c. During the period from October 1 to October 16, there were about four new cases per day. Over the 8-day period from October 1 to October 9, that gives an expected increase of 8×4 cases. Using this result and the fact that there were 28,408 cases by October 1, we can estimate the total for October 9 as

$$\begin{aligned}\text{Estimated cumulative cases by October 9} &= \text{Number of cases by October 1} + \text{Number of new cases}\\ N(8) &\approx N(0) + 8 \times \text{Average rate of change}\\ &= 28,408 + 8 \times 4\\ &= 28,440\end{aligned}$$

This estimate is illustrated in **Figure 1.17**, where the point (8, 28,440) lies on the line joining the nearest two data points.

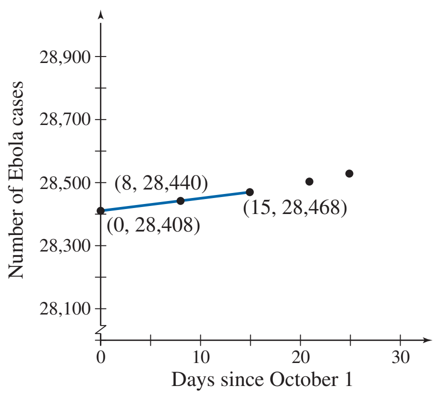


Figure 1.17 Ebola data: The estimate for October 9 lies on the line joining data from October 1 and October 16.

It turns out that the actual number reported by October 9, 2015 was 28,429. Our estimate using the average rate of change got us pretty close to the right number.

TRY IT YOURSELF 1.13 Brief answers provided at the end of the section.

Calculate the average rate of change from October 22 to October 26. Use your answer to estimate the cumulative number of cases reported by October 25.

EXAMPLE 1.14 Interpreting the Average Rate of Change: Newton's Law of Cooling

Newton's law of cooling describes how the temperature of an object changes over time. Suppose an aluminum pan is removed from an oven and left to cool in a room where the air temperature is constant. The table gives the temperature $A(t)$ in degrees Fahrenheit of the aluminum as a function of the time t in minutes since it was removed from the oven. The plot of the data shown in **Figure 1.18** illustrates the rapid initial decrease in temperature.

t = Time in minutes	A = Temperature in degrees Fahrenheit
0	300
30	150
60	100
90	79
120	73
150	72
180	71
210	71

a. Calculate the average rate of change from $t = 0$ to $t = 30$. Explain the meaning of the number you calculated in terms of the temperature of the aluminum.

b. Calculate the average rate of change from $t = 90$ to $t = 120$.

c. Use the results in parts a and b to answer the following question: Does a hot object cool more rapidly when the difference between its temperature and that of the air is larger or when that difference is smaller?

d. Use the average rate of change to estimate the temperature of the aluminum 100 minutes after it is removed from the oven.

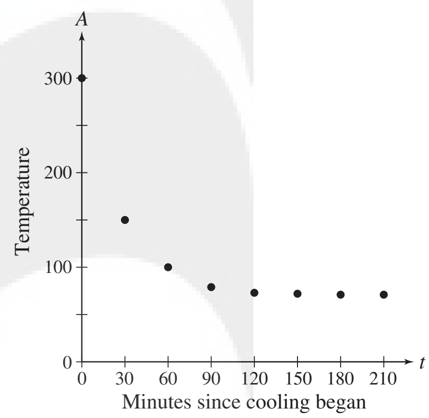


Figure 1.18 Plot of temperature data

SOLUTION

a. The change in temperature over the first 30 minutes is

$$A(30) - A(0) = 150 - 300 = -150°$$

The negative sign reflects the fact that the temperature is decreasing. This drop occurred over a 30-minute period, so

$$\text{Average rate of change} = \frac{-150}{30} = -5° \text{ per minute}$$

This result means that over the first 30 minutes, the temperature is decreasing by an average of 5° each minute.

b. We calculate the same way as in part a:

$$\begin{aligned}\text{Average rate of change} &= \frac{A(120) - A(90)}{120 - 90}\\ &= \frac{73 - 79}{120 - 90}\\ &= \frac{-6}{30}\\ &= -0.2^\circ \text{ per minute}\end{aligned}$$

c. The temperature drops at an average rate of 5° per minute during the time the aluminum is hotter, and it drops at an average rate of only 0.2° per minute when the temperature of the aluminum is nearer that of the surrounding air. As we might expect, a larger temperature difference means more rapid cooling.

d. Between $t = 90$ and $t = 120$, the aluminum cools at an average rate of 0.2° per minute. At 90 minutes the temperature is 79°. We estimate its temperature 10 minutes later using

$$\text{Temperature at 100 min.} \approx \text{Temperature at 90 min.} + 10 \times \text{Average rate of change}$$

$$\begin{aligned}A(100) &\approx A(90) + 10 \times \text{Average rate of change}\\ &= 79 + 10 \times (-0.2)\\ &= 77^\circ\end{aligned}$$

This gives an estimated temperature of 77° after 100 minutes of cooling. **Figure 1.19** shows that the temperature estimate for 100 minutes is on the line joining the data points for 90 and 120 minutes.

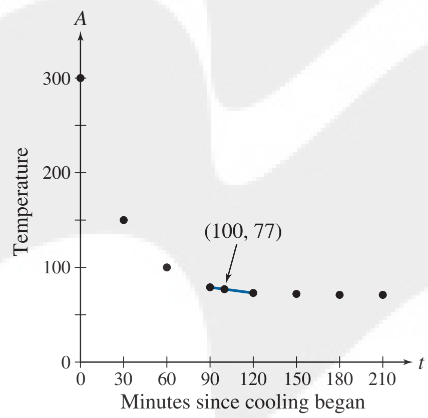


Figure 1.19 Temperature data: The point (100, 77) lies on the line joining the nearest two data points.

TRY IT YOURSELF 1.14 Brief answers provided at the end of the section.

Use the average rate of change in temperature from $t = 30$ to $t = 60$ to estimate the temperature at $t = 35$. Round your answer to the nearest degree.

EXTEND YOUR REACH

In the example, what can you say about the average rate of change in temperature over a single minute after the aluminum has cooled for a very long time?

TRY IT YOURSELF ANSWERS

1.10 $-\frac{1}{3}$

1.11 Simplify $\dfrac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a}$.

1.12 93 ants per week

1.13 The average rate of change is six new cases per day. That gives an estimate of 28,522 cases reported by October 25.

1.14 About 142°

EXERCISE SET 1.2

CHECK YOUR UNDERSTANDING

1. **True or false:** The value of the average rate of change depends both on the function and on the interval over which it is calculated.
2. For a function f, the expression $\dfrac{f(7)-f(4)}{3}$ gives ____________.
3. The line that is related to the average rate of change is called ____________.
4. **True or false:** The average rate of change is always positive.
5. If $f(a) < f(b)$, then the average rate of change of f over the interval $[a, b]$:
   a. is positive
   b. is negative
   c. is zero
   d. cannot be determined without further information
6. If the average rate of change of f on the interval $[a, b]$ is 0, then which is true?
   a. $f(a) > f(b)$
   b. $f(a) < f(b)$
   c. $f(a) = f(b)$
   d. The relationship between $f(a)$ and $f(b)$ cannot be determined without further information.
7. The average rate of change can be calculated for an interval of any positive length. Can it be calculated for an interval of length 0?
8. What is the average rate of change on any interval of a constant function?

SKILL BUILDING

Calculating average rates of change. In Exercises 9 through 19, calculate the average rate of change indicated.

9. $f(x) = 2x + 1$ from $x = 1$ to $x = 4$
10. $f(x) = x^2$ from $x = 2$ to $x = 5$
11. $f(x) = \dfrac{1}{x}$ from $x = 1$ to $x = 2$
12. $f(x) = x^3 + x + 1$ on $[2, 3]$
13. $f(x) = \dfrac{x+1}{x-1}$ on $[3, 5]$
14. $f(x) = \sqrt{x}$ on $[0, 10]$
15. $f(x) = \dfrac{1}{\sqrt{x}}$ on $[1, 4]$
16. $f(x) = x - x^2$ on $[0, 1]$
17. $f(x) = x^4 + x^2$ on $[-2, 2]$
18. $f(x) = \dfrac{1}{2x+1}$ on $[3, 4]$
19. $f(x) = |x|$ on $[-3, -2]$

Units associated with average rate of change and their meaning. In Exercises 20 through 30, state the units associated with the average rate of change of the given function, and explain what the average rate of change means in terms of the variables of the exercise.

20. $F(d)$ is the cumulative number of flu cases reported as of day d.
21. $B(t)$ is the balance in dollars of a savings account after t months.
22. $A(d)$ is the cumulative number of traffic accidents at a certain intersection as of day d.
23. $M(h)$ is the total miles driven after h hours.

24. $W(y)$ is the weight in pounds of a growing child who is y years old.
25. $V(t)$ is the speed in miles per hour of an airplane t minutes after takeoff.
26. $T(I)$ is your income tax liability in dollars if your income is I dollars.
27. $P(n)$ is the profit in dollars your company earns if you produce n items.
28. $H(s)$ is the heart rate in beats per minute of an athlete who is running at s feet per second.
29. $A(d)$ is the area in square feet of a circle of diameter d measured in feet.
30. $E(m)$ is the elevation in feet along a hiking trail m miles from its beginning.

PROBLEMS

Average rates of change over general intervals. In Exercises 31 through 42, calculate the average rate of change of the given function on the interval $[a, a+h]$ (assuming $h > 0$). Simplify your answers as much as possible.

31. $f(x) = c$, where c is a constant.
32. $f(x) = x$
33. $f(x) = 2 - 5x$
34. $f(x) = mx + b$, where m and b are constants.
35. $f(x) = x^2$
36. $f(x) = 3x^2 + 4x$
37. $f(x) = 2 - x^2$
38. $f(x) = \sqrt{x+1}$ (Assume $a \geq -1$.)
39. $f(x) = 2\sqrt{x}$ (Assume $a \geq 0$.)
40. $f(x) = x^3$ *Suggestion*: Recall that $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.
41. $f(x) = \dfrac{1}{x}$ (Assume $a > 0$.)
42. $f(x) = \dfrac{1}{\sqrt{x}}$ (Assume $a > 0$.) *Suggestion*: If you have difficulty, look at Example 1.11.
43. **Constants and the average rate of change.** How does the average rate of change of $f(x) + c$ compare with the average rate of change of f?

Estimating unknown function values. In Exercises 44 through 50, use the following information. If $a + h$ lies in the interval $(a, b]$ and $f(a)$ and $f(b)$ are known, then we can use the average rate of change to approximate $f(a+h)$:

$$f(a+h) \approx f(a) + h\frac{f(b) - f(a)}{b - a}$$

You will be led to a derivation of this formula in a later exercise showing that the formula is using a secant line to approximate the graph.

44. For $f(x) = \sqrt{x}$, use the values of $f(100)$ and $f(121)$ to estimate the value of $\sqrt{107}$.
45. For $f(x) = \sqrt[3]{x}$, use the values of $f(8)$ and $f(27)$ to estimate the value of $\sqrt[3]{20}$.
46. For $f(x) = \sqrt{x}$, use the values of $f(25)$ and $f(36)$ to estimate the value of $\sqrt{30}$.
47. For $f(x) = 2^x$, use the values of $f(2)$ and $f(3)$ to estimate the value of $2^{2.4}$.
48. Suppose that $f(x)$ is a function with $f(4) = 3$ and $f(7) = 4$. Estimate $f(5)$.
49. Suppose that $f(x)$ is a function with $f(3) = 9$ and $f(7) = 5$. Estimate $f(5)$.
50. **Using different intervals.**
    - **a.** Use an average rate of change over the interval $[4, 25]$ to estimate the value of $\sqrt{7}$.
    - **b.** Use an average rate of change over the interval $[4, 9]$ to estimate the value of $\sqrt{7}$.
    - **c.** Which of the two preceding estimates is nearer the actual value of $\sqrt{7}$?
    - **d.** Based on the result of this exercise, do you think better results are obtained when using shorter or longer intervals?
51. **Average rate of change for the absolute value function.** Let $f(x) = |x|$. Show that if $0 < a < b$, then the average rate of change of f on $[a, b]$ is 1. Show that if $a < b < 0$, then the average rate of change of f on $[a, b]$ is -1. Show that if $a > 0$, then the average rate of change of f on $[-a, a]$ is 0.

MODELS AND APPLICATIONS

52. Carbon-14. Carbon-14 is a radioactive substance that decays over time. It is used to date fossils and other objects containing organic matter. In the following table, $C(t)$ is the amount in grams of carbon-14 remaining at time t, measured in thousands of years.

t = time in thousands of years	0	10	15	20
C = amount in grams remaining	5	1.5	0.8	0.4

a. Use the average rate of change to estimate the amount of carbon-14 remaining after 12,000 years. Be careful about the units.

b. We can reverse the roles of the variables and think of t as depending on C. Call this function g. For example, $g(0.8) = 15$ because $t = 15$ when $C = 0.8$.

i. Use the table to calculate the average rate of change for g from 0.8 gram to 1.5 grams.

ii. Use your answer to the preceding part to estimate the time when there are 1.3 grams remaining.

53. Prescription drugs. The following table shows the number N (in billions) of prescriptions filled in the United States in the given year.[2]

Year	2007	2013	2014	2015
Number of prescriptions	3.50	3.99	4.08	4.17

a. Make a table showing, for each of the periods in the table, the average rate of change per year in the number of prescriptions.

b. During which of these periods was the number of prescriptions growing at its smallest rate?

c. Use the average rate of change from 2014 to 2015 to estimate the number of prescriptions in 2020.

54. Arterial blood flow. A relatively small increase in the radius of an artery corresponds to a relatively large increase in blood flow. For example, if one artery has a radius 5% larger than another, the blood flow rate is 1.22 times as large. The accompanying table gives more information.

Increase in radius	5%	10%	15%	20%
Times greater blood flow rate	1.22	1.46	1.75	2.07

Use the average rate of change to estimate how many times greater the blood flow rate is in an artery that has a radius 7% larger than another.

55. Belk revenues. The retailer Belk's annual reports show the following revenues in billions of dollars.

Year	2009	2012	2015	2018
Revenue	3.50	3.70	4.11	3.60

a. During which 3-year period was the average rate of change in revenue negative?

b. Calculate the average rate of change in revenue from 2015 to 2018. Be sure to give proper units.

c. Use your answer from part b to give an estimated revenue for 2020.

56. Scooter rides. The number N of scooter rides in a city t weeks after the launch of a scooter ride business is given by

$$N = \frac{12{,}500}{1 + 20 \times 0.87^t}$$

Round values of N to the nearest integer.

a. Calculate the average rate of change per week in rides for

- Week 50 to week 55
- Week 55 to week 60
- Week 60 to week 65

b. As of week 65, you are considering investing in the scooter business. The profitability of the investment depends on the growth in scooter rides. Do the calculations from part a indicate that this is a wise investment?

57. Choosing a bat. A 26-inch baseball bat is recommended for a child who is 38 inches tall. As the child grows, up to 62 inches, the average rate of change in recommended bat length is 0.25 inch of bat length per inch of growth. Bat lengths are in whole numbers of inches. How many bats will you buy from the time your child is 38 inches tall through the time your child is 62 inches tall?

[2]See http://www.statista.com/statistics/261303/total-number-of-retail-prescriptions-filled-annually-in-the-us.

58. Yellowfin tuna. The following table shows data from a study of yellowfin tuna.[3]

L = length (centimeters)	100	110	120	130	140	160
W = weight (pounds)	42.5	56.8	74.1	94.7	119.0	179.0

a. What is the average rate of change, in pounds per centimeter, for lengths from 100 to 110 centimeters? From 140 to 160 centimeters?

b. Two fishermen tell tales slightly exaggerating the lengths of a yellowfin tuna they caught, with one claiming to have caught a 100-centimeter fish and the other a 150-centimeter fish. Which fish tale indicates the larger exaggeration in weight?

c. Estimate the weight of a 108-centimeter yellowfin tuna.

d. Estimate the length of a yellowfin tuna weighing 155 pounds.

59. Puppy weight. A woman wants to estimate the adult weight of her puppy based on the age at which it reaches a critical weight. She knows that puppies reaching that critical weight at age 5 weeks will weigh 52 pounds as adults, and puppies reaching that critical weight at age 12 weeks will weigh 22 pounds as adults. If her puppy reaches that critical weight at age 7 weeks, estimate the adult weight of her puppy. Round your answer to the nearest pound.

60. Home equity. When a bank loans money for a home, it effectively owns the home. Each payment you make is partly interest and partly purchase price for a small share in home ownership. The ownership share that you have purchased is your equity. If you secure a 30-year home mortgage for \$350,000 at an annual percentage rate, or APR, of 6%, then after k monthly payments your equity is

$$E = \frac{350{,}000(1.005^k - 1)}{1.005^{360} - 1} \text{ dollars}$$

a. Do you own half the home after half of the payments (that is, 180 payments) have been made?

b. Calculate the average rate of change per month for the following periods.

- From $k = 0$ to $k = 12$ (the first year of payments)
- From $k = 108$ to $k = 120$ (the 10th year)
- From $k = 228$ to $k = 240$ (the 20th year)
- From $k = 348$ to $k = 360$ (the 30th year)

c. Does your equity grow faster early or late in the term of the loan?

61. Spending on video games. The following table shows the amount, in billions of dollars, spent on video game content in the United States in the given year.[4]

Year	2013	2014	2015	2016	2017
Spent (billions of dollars)	20.2	21.4	23.2	24.5	29.1

a. Make a table showing the average rate of change *per month* for each period shown in the table. Round your answers to two decimal places.

b. Over which period is the average rate of change the largest?

62. A falling rock. A rock dropped near the surface of Earth falls $D(t) = 16t^2$ feet in t seconds. Figure 1.20 shows how the distance fallen varies with time. The aim of this exercise is to determine the speed of the rock at $t = 2$ seconds—a determination that is more difficult than it appears at first sight. We calculate the speed over a time interval using the average rate of change for the function D:

$$\text{Speed} = \frac{\text{Distance traveled}}{\text{Elapsed time}}$$

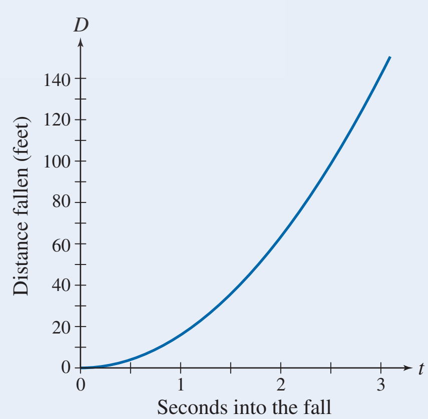


Figure 1.20 Distance fallen by a rock

[3] *Mathematics for the Biosciences,* Michael R. Cullen, TechBooks, Fairfax, VA, 1983.

[4] See http://www.theesa.com/about-esa/essential-facts-computer-video-game-industry.

We ask: How is it possible to find the speed when the elapsed time is 0?

a. Calculate the speed over the following time intervals, and report your answers correct to four decimal places.

i. From $t = 2$ to $t = 3$ **ii.** From $t = 2$ to $t = 2.5$

iii. From $t = 2$ to $t = 2.1$ **iv.** From $t = 2$ to $t = 2.01$

v. From $t = 2$ to $t = 2.001$ **vi.** From $t = 2$ to $t = 2.0001$

b. Without calculating further, and based on your calculations in part a, what do you think the speed at $t = 2$ is?

CHALLENGE EXERCISES FOR INDIVIDUALS OR GROUPS

63. A formula for estimating unknown function values. In this exercise, you are led to derive a formula for approximating unknown function values in terms of the average rate of change. In the process, you will see that the formula is using a secant line to approximate the graph. Suppose we know the values of $f(a)$ and $f(b)$, but for some $a+h$ in the interval $[a, b]$ the function value $f(a+h)$ is unknown.

a. Use the fact that the slopes of the lines in **Figure 1.21** are nearly the same to obtain the approximate equality

$$\frac{f(a+h)-f(a)}{h} \approx \frac{f(b)-f(a)}{b-a}$$

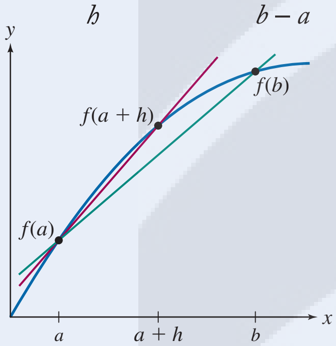


Figure 1.21 Two lines with nearly the same slopes

b. Use the results from part a to obtain the approximate equality

$$f(a+h) \approx f(a) + h\frac{f(b)-f(a)}{b-a}$$

This approximation is illustrated in **Figure 1.22**. Note that we are approximating the graph of f by the secant line.

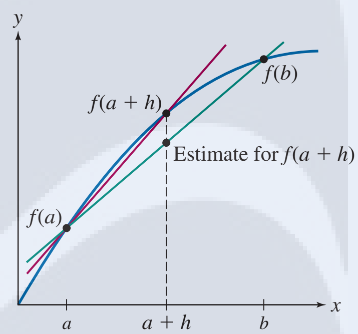


Figure 1.22 Approximate value for $f(a+h)$

REVIEW AND REFRESH: Exercises from Previous Sections

64. From Section P.1: What is the distance from $(-3, 5)$ to $(13, 5)$?

Answer: 16

65. From Section P.2: Solve the inequality $x-4 < 3x+6$.

Answer: $(-5, \infty)$

66. From Section P.3: Solve the inequality $x^2 < 3x$.

Answer: $(0, 3)$

67. From Section 1.1: If $f(x) = \dfrac{x+1}{x}$, find $f(x+1)$.

Answer: $\dfrac{x+2}{x+1}$

68. From Section 1.1: If $f(x) = x^2 - 1$, find $f(\sqrt{x})$.

Answer: $x-1$

69. From Section 1.1: Find the domain of $f(x) = \dfrac{x+1}{\sqrt{x-1}}$.

Answer: $(1, \infty)$

70. From Section 1.1: Find the domain of $f(x) = \dfrac{x}{x^{1/4}}$.

Answer: $(0, \infty)$

71. From Section 1.1: Find the range of $g(t) = t^4 + 7$.

Answer: $[7, \infty)$

1.3 Graphs and Rates of Change

Rates of change are the key to a dynamic view of functions.

1.1 The Basics of Functions
1.2 Average Rates of Change
1.3 Graphs and Rates of Change
1.4 Limits and End Behavior of Graphs

In this section, you will learn to:

1. Calculate the rate of change of a function at a point given information about the tangent line.
2. Locate the intervals on which a function is increasing or decreasing.
3. Relate the intervals on which a function is increasing or decreasing to the sign of the rate of change.
4. Identify local and absolute maximum and minimum values of a function.
5. Locate the intervals on which a graph is concave up or concave down.
6. Interpret concavity in terms of rates of change.
7. Identify the inflection points of a graph.
8. Sketch graphs having specified properties.
9. Interpret the shape of a graph in applications.

A familiar formula for calculating velocity is

$$\text{Velocity} = \frac{\text{Distance traveled}}{\text{Elapsed time}}$$

This formula gives the average velocity, the average rate of change in distance with respect to time. But the speedometer in your car (pictured in Figure 1.23) shows a different type of velocity. If you accelerate on a narrow road to pass a slow-moving truck, the speedometer needle might move smoothly from 40 to 65 miles per hour and just as smoothly slow to 60 miles per hour. As the needle moves, it shows your velocity at each instant of time: your instantaneous velocity.

Figure 1.23 Car speedometer: The speedometer in your car shows instantaneous velocity. Jamesbowyer/Getty Images

The same idea applies to any function. We can calculate an average rate of change, but there is also the instantaneous rate of change or just the rate of change. Just as the speedometer in your car provides important information regarding your driving, the instantaneous rate of change provides invaluable information regarding the function.

The rate of change of a function is a central idea in calculus and its applications, and for many good reasons. Among them is the fact that the rate of change of a function describing a natural phenomenon is often easier to understand than the phenomenon itself. For example, if a rock is dropped, its velocity as a function of time is difficult to measure. But the rate of change in the velocity of the rock, which is the acceleration due to gravity, is easier to measure because it is constant. Calculus then shows us how to use acceleration to find velocity.

In this section, we focus on graphs and their relation to the rate of change. The graph can reveal how functions change, show where functions increase or decrease, and identify maximum and minimum values of a function. Dynamic properties such as increase, decrease, and concavity provide a preview of concepts that will be important in calculus.

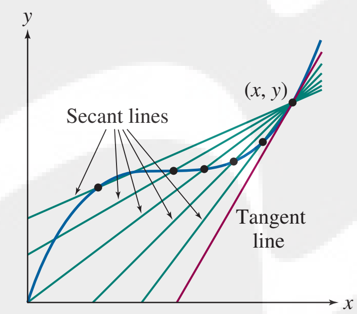
An Intuitive Description of Rates of Change

The rate of change is the slope of a line tangent to the graph.

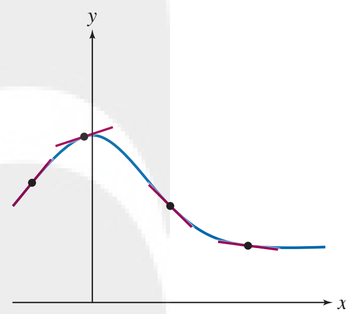
The **rate of change** at x of the function f is the slope of the tangent line to the graph $y = f(x)$ at the point $(x, f(x))$.

In **Figure 1.24**, we have plotted secant lines for the graph of a function $y = f(x)$ through the point (x, y) and over ever shorter intervals. Note how these secant lines approach a line that appears to be tangent to the curve (similar to the way a line may be tangent to a circle). This line is aptly named the tangent line to the graph at (x, y). Just as the secant lines approach the tangent line, the slopes of the secant lines, the average rates of change, approach the slope of the tangent line. The slope of the tangent line is the instantaneous rate of change or simply **rate of change**[5] at x of the function f. (In calculus the rate of change is called the derivative.)

In **Figure 1.25**, we have plotted a number of tangent lines to the graph of a function $y = g(x)$. The slopes of these tangent lines give the rate of change of the function at the corresponding x-value. Note how the tangent lines track the path of the graph. That is, tangent lines with a positive slope indicate a rising graph, and tangent lines with a negative slope indicate a falling graph. Note also that steeper tangent lines correspond to steeper sections of the graph. Thus, we can associate the rate of change at a point x with the "direction" of the graph of g at the corresponding point.



DF **Figure 1.24** Secant lines and tangent lines: Over ever shorter intervals, the secant lines approach the tangent line.



DF **Figure 1.25** Tangent lines: These superimposed lines track the path of the graph, indicating its direction.

CONCEPTS TO REMEMBER: Properties of the Rate of Change of a Function

- Secant lines for the graph through a given point on the graph and over ever shorter intervals approach the tangent line, as is illustrated in Figure 1.24.
- The slope of the tangent line is the instantaneous rate of change or just the rate of change of the function at the given value of the independent variable.
- The average rate of change gives an approximation to the (instantaneous) rate of change. Average rates of change over shorter and shorter intervals give better and better approximations of the rate of change.
- Positive rates of change indicate regions where the graph is rising.
- Negative rates of change indicate regions where the graph is falling.
- Steeper tangent lines correspond to steeper portions of the graph.

[5]Not all functions have a well-defined rate of change at every point. But for the classes of functions we will consider later (linear, exponential, etc.), the rate of change is defined at each point in the domain. The discussion in this section is intuitive, and we assume that the rate of change exists.

EXAMPLE 1.15 Finding a Rate of Change

The graph of $f(x) = x^2$ and the tangent line to the graph for $x = 1$ are shown in **Figure 1.26**. The tangent line also passes through the point (2, 3). Find the rate of change of f at $x = 1$.

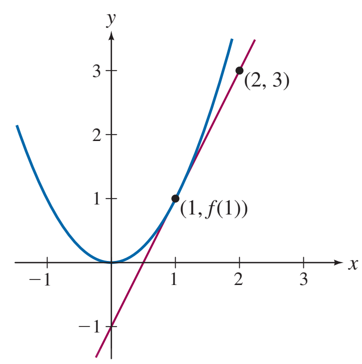


Figure 1.26 The graph of $f(x) = x^2$ and its tangent line for $x = 1$

SOLUTION

The rate of change of f at $x = 1$ is the slope of the tangent line. The tangent line passes through the points $(1, f(1)) = (1, 1)$ and (2, 3). Recall how we use these two points to calculate the slope:

$$\begin{aligned}\text{Rate of change} &= \text{Slope of tangent line}\\ &= \frac{\Delta y}{\Delta x}\\ &= \frac{3-1}{2-1}\\ &= 2\end{aligned}$$

Thus, the rate of change of f at $x = 1$ is 2.

TRY IT YOURSELF 1.15 Brief answers provided at the end of the section.

The tangent line to the graph of $f(x) = x^2$ at $x = 2$ passes through the points $(2, f(2))$ and (3, 8). Find the rate of change of f at $x = 2$.

Increasing and Decreasing Functions

Rates of change can show where functions are increasing and where they are decreasing.

Where the graph of a function is rising, it is natural to think of the function as an **increasing function** there. Similarly, where the graph falls, we think of the function as a **decreasing function** there.

In words, a function $f(x)$ is increasing if larger values of x yield larger values of f. Similarly, it is decreasing if larger values of x yield smaller values of f.

Regions of increase or decrease for a function may not be apparent from a formula, but they can be seen from a graph. For example, it is by no means apparent where the function $f(x) = \dfrac{x^3 - 48x}{128}$ is increasing or where it is decreasing. We show the graph in

An **increasing function** on an interval I is a function f such that $f(x) < f(y)$ whenever $x < y$ and both x and y are in the interval I.

A **decreasing function** on an interval I is a function f such that $f(x) > f(y)$ whenever $x < y$ and both x and y are in the interval I.

Figure 1.27. In Figure 1.28, we have marked the regions where the graph rises and where it falls. We conclude that f is increasing on $(-\infty, -4]$ and $[4, \infty)$ and is decreasing on $[-4, 4]$.

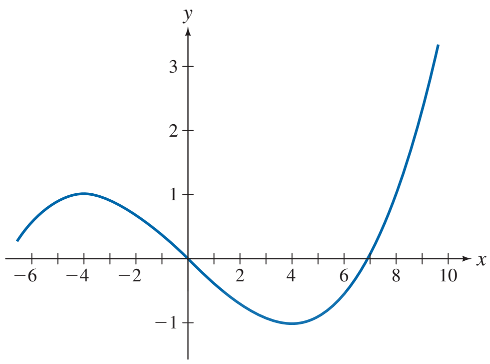
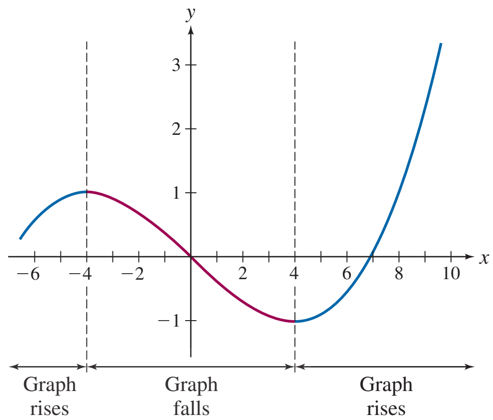


Figure 1.27 The graph of $f(x) = \dfrac{x^3 - 48x}{128}$



DF Figure 1.28 The graph of $f(x) = \dfrac{x^3 - 48x}{128}$: Regions where the graph rises and where it falls are marked.

In Figure 1.29, we have added a few tangent lines. We see that their slopes, the rates of change, give additional information. The figure shows that positive rates of change indicate regions of increase, and negative rates of change indicate regions of decrease.

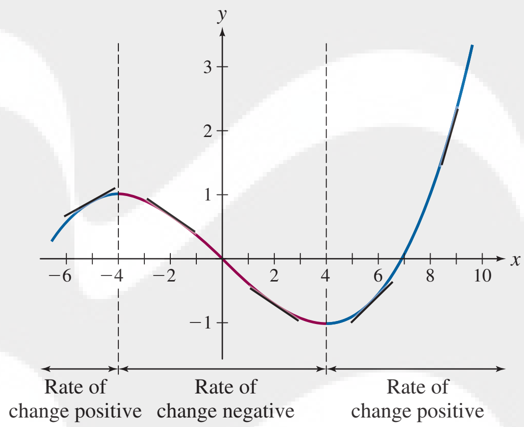


Figure 1.29 The graph of $f(x) = \dfrac{x^3 - 48x}{128}$: Positive rates of change indicate regions of increase, and negative rates of change indicate regions of decrease.

CONCEPTS TO REMEMBER: How Rates of Change Determine Increase or Decrease

- A positive rate of change indicates a rising graph and thus a region of increase for a function. Conversely, if a function is increasing on an interval, then the rate of change is nonnegative there.
- A negative rate of change indicates a falling graph and thus a region of decrease for a function. Conversely, if a function is decreasing on an interval, then the rate of change is nonpositive there.
- Rates of change that are larger in magnitude indicate a steeper graph.

EXAMPLE 1.16 Sketching a Graph

Sketch the graph of a function that has a positive rate of change on $(-\infty, 0)$ and a negative rate of change on $(0, \infty)$.

SOLUTION

Many graphs satisfy the given information, so there are many correct answers. We provide one here. The rate of change is used to determine where the graph rises and where it falls.

- The function has a positive rate of change on $(-\infty, 0)$. Consequently, the function is increasing, and the graph rises on this interval. This fact shows us how to begin the graph, as we have done in **Figure 1.30**.
- The function has a negative rate of change on $(0, \infty)$. Consequently, the function is decreasing, and the graph falls on this interval. This fact allows us to complete the graph as shown in **Figure 1.31**.

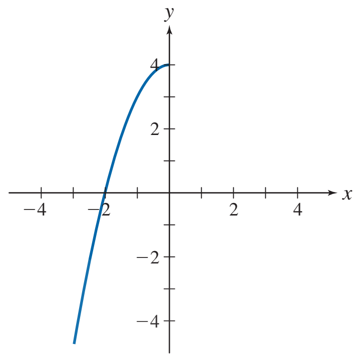


Figure 1.30 Positive rate of change: The graph rises on $(-\infty, 0)$.

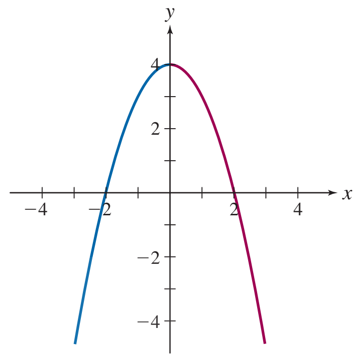


Figure 1.31 Negative rate of change: The fall on $(0, \infty)$ completes the graph.

TRY IT YOURSELF 1.16 Brief answers provided at the end of the section.

Sketch the graph of a function with a negative rate of change on $(-\infty, 0)$ and a positive rate of change on $(0, \infty)$.

Maximum and Minimum Values

Functions may reach maximum or minimum values where the rate of change is zero.

Let's examine the graph of $f(x) = \dfrac{x^3 - 48x}{128}$ further. Other interesting features of the graph include the peak at $x = -4$ and the valley at $x = 4$, which are shown in **Figure 1.32**.

The peak at $x = -4$ corresponds to a **local maximum** value of $y = 1$ for f, and the valley at $x = 4$ corresponds to a **local minimum** value of $y = -1$.

We use the term *local* here because the peak at $x = -4$ is not the highest point on the entire graph over $\mathbb{R}$—it is only the highest among nearby points. Likewise, the valley at $x = 4$ is not the lowest point on the entire graph—it is only the lowest among nearby points. The terms *relative maximum* and *relative minimum* are also used.

In **Figure 1.33**, we have added tangent lines at both the local maximum and the local minimum. Both of these lines are horizontal and so have slope 0. This illustrates

A function f reaches a **local maximum** value of $f(x_0)$ at $x = x_0$ if there is an interval (a, b) containing x_0 so that $f(x_0) \geq f(x)$ for every x in the interval (a, b).

A function f reaches a **local minimum** value of $f(x_0)$ at $x = x_0$ if there is an interval (a, b) containing x_0 so that $f(x_0) \leq f(x)$ for every x in the interval (a, b).

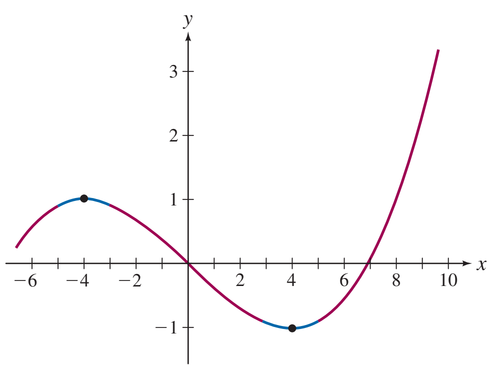


Figure 1.32 Peaks and valleys: The peak at $x = -4$ is the highest among nearby points (blue) on the graph. The valley at $x = 4$ is the lowest among nearby points (blue) on the graph.

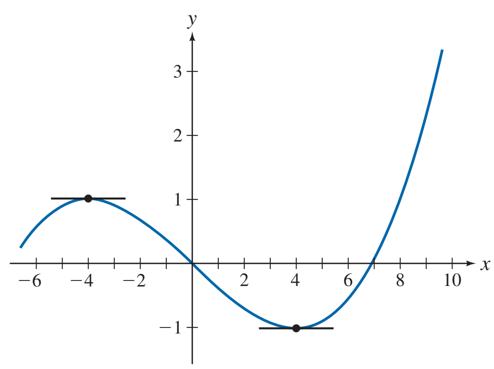


Figure 1.33 Tangent lines added: The rate of change is typically 0 at local maxima and minima.

the fact that at local maxima (plural of maximum) or minima, the rate of change is typically 0.

If there is a highest point on the entire graph, we say that point represents an **absolute maximum** value. Similarly, the **absolute minimum** value, if it exists, is indicated by the lowest point on the entire graph. Such points may occur at local maxima or minima, or at endpoints of graphs.

A function f reaches an **absolute maximum** value of $f(x_0)$ at $x = x_0$ if $f(x_0) \geq f(x)$ for every x in the domain of f.

A function f reaches an **absolute minimum** value of $f(x_0)$ at $x = x_0$ if $f(x_0) \leq f(x)$ for every x in the domain of f.

For example, the function $f(x) = \dfrac{x^3 - 48x}{128}$ has neither an absolute maximum nor an absolute minimum. But if we restrict the domain of f to the interval $[-6, 10]$ shown by the graph in Figure 1.32, then it has both. The function with a restricted domain has an absolute minimum value of $y = -1$, which occurs at $x = 4$. It has an absolute maximum value of $y = 4.0625$, which occurs at the endpoint $x = 10$.

Note the difference between a local maximum value and the place where it occurs. In the preceding example $f(x) = \dfrac{x^3 - 48x}{128}$, the local maximum value is $f(-4) = 1$, and it occurs at $x = -4$. Note also that the definitions are structured so that neither a local maximum nor a local minimum can occur at an endpoint of a graph.

EXAMPLE 1.17 Identifying Features of a Graph

Figure 1.34 shows the graph of a function $y = f(x)$ whose domain is $[-3, 3]$.

a. Identify the intervals of increase and decrease.

b. Identify the regions where the rate of change is positive and the regions where it is negative. (Do not include the endpoints of the domain.)

c. Determine any local maxima or minima. What is the rate of change at these points?

d. Find any absolute maxima or minima.

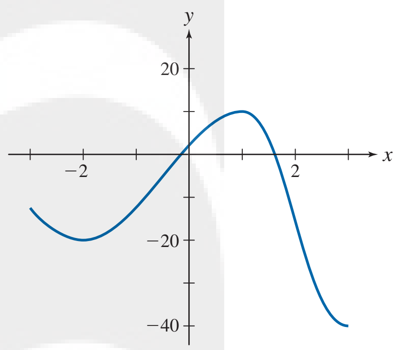


Figure 1.34 Graph for Example 1.17: The domain is $[-3, 3]$.

SOLUTION

a. As we move from left to right, the graph falls until $x = -2$ and after $x = 1$. Hence, f decreases on $[-3, -2]$ and on $[1, 3]$. The graph rises after $x = -2$ and before $x = 1$, so f increases on the interval $[-2, 1]$. These features of the graph are shown in Figure 1.35.

b. The rate of change is positive on intervals where the graph is rising, in this case $(-2, 1)$. The rate of change is negative on intervals where the graph is falling, in this case $(-3, -2)$ and $(1, 3)$.

c. The graph shows that f has a local minimum value of $y = -20$ at $x = -2$ and that f has a local maximum value of $y = 10$ at $x = 1$. The rate of change is 0 at both of these values of x.

d. The highest point on the graph occurs at $x = 1$, and f has an absolute maximum value of $y = 10$ there. The lowest point on the graph is at the right endpoint. Thus, the absolute minimum of f occurs at $x = 3$, and the value is $y = -40$.

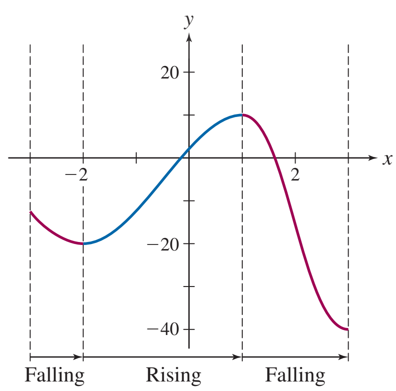


Figure 1.35 Rising and falling: For part a, we mark the intervals of increase and decrease.

TRY IT YOURSELF 1.17 Brief answers provided at the end of the section.

Figure 1.36 shows the graph of a function g whose domain is $[-3, 3]$. Identify the intervals of increase and decrease, and determine any local maxima or minima.

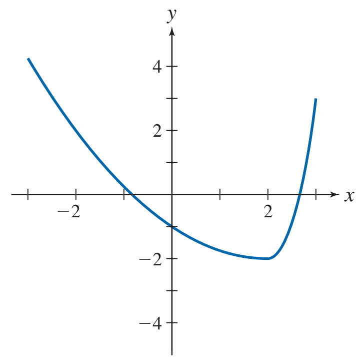


Figure 1.36 Graph for Try It Yourself 1.17: The domain is $[-3, 3]$.

EXAMPLE 1.18 Showing Maxima and Minima on a Graph

Sketch the graph of a function $y = f(x)$ with the following properties:

a. The domain of f is $[-3, 5]$.

b. The rate of change of f is negative on the intervals $(-3, -2)$ and $(2, 5)$, and f is increasing elsewhere.

c. f has an absolute maximum at $x = 2$.

d. f has a local minimum at $x = -2$, but this is not the location of an absolute minimum.

SOLUTION

We show how each of the preceding properties contributes to the construction of the graph.

a. We use the domain $[-3, 5]$ to determine the horizontal extent of the graph.

b. We know that f is decreasing on the intervals $(-3, -2)$ and $(2, 5)$ where the rate of change is negative, and it is increasing on $(-2, 2)$. The intervals of increase and decrease tell us how to start the graph. It falls from $x = -3$ to $x = -2$, rises from $x = -2$ to $x = 2$, and starts to fall again after that. These features are indicated in Figure 1.37.

c. We want to be sure that the highest point on the graph occurs at $x = 2$ because that is the location of the absolute maximum.

d. There will be a local minimum at $x = -2$, but because this is not supposed to be the location of an absolute minimum, we must be sure that it is not the lowest point on the entire graph. We can arrange that by making the right endpoint the absolute minimum.

The completed graph is shown in Figure 1.38, where we have also marked both local and absolute maxima and minima.

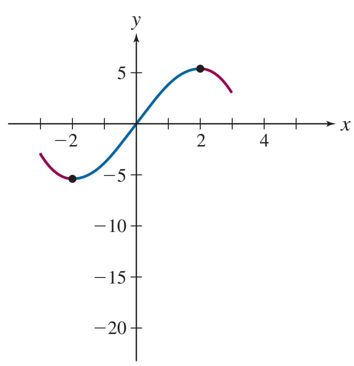


Figure 1.37 Beginning the graph: Regions of increase and decrease are indicated.

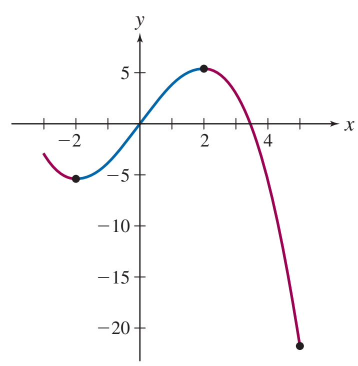


Figure 1.38 Completing the graph: Ensure an absolute maximum at $x = 2$ and an absolute minimum at $x = 5$.

TRY IT YOURSELF 1.18 Brief answers provided at the end of the section.

Sketch the graph of a function f with the following properties. The domain of f is $[-4, 4]$, and f decreases from $x = -4$ to $x = 0$, where it reaches both a local minimum and an absolute minimum. There is an absolute maximum at each endpoint. There is no local maximum.

EXTEND YOUR REACH

a. Is it possible to make the graph of a function with the following properties? Its domain is the set of all real numbers, it is increasing on $(-\infty, 0)$ and has a negative rate of change on $(0, \infty)$, but it has no local maximum. *Suggestion*: The graph you make may have breaks.

b. Is it possible to make the graph of a function that has a zero rate of change at $x = 0$ but reaches neither a maximum nor a minimum at $x = 0$? *Suggestion*: Sketch the tangent line to the graph of $y = x^3$ at $x = 0$.

Concavity and Rates of Change

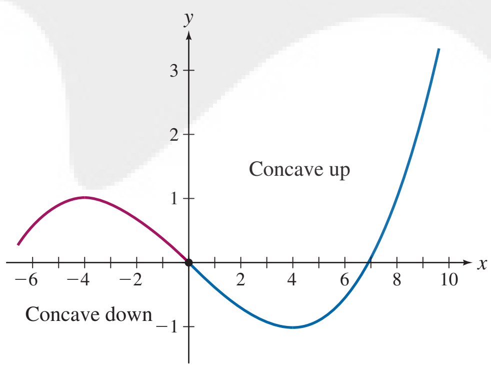
Concavity describes the shape of a graph and can be characterized in terms of rates of change.

A graph of a function is **concave up** if it looks like a piece of wire whose ends are bent upward.

A graph of a function is **concave down** if it looks like a piece of wire whose ends are bent downward.

An **inflection point** on the graph of a function is a point where the direction of bending changes.

Let's focus once more on the graph of $f(x) = \dfrac{x^3 - 48x}{128}$, as shown in **Figure 1.39**. The part of the graph to the left of $x = 0$ has the shape of a piece of wire whose ends are bent downward. To the right of $x = 0$, the graph has the shape of a wire whose ends are bent upward. At $x = 0$, the direction of bending changes. This concept of direction of bending is important enough to have its own terminology: **concave up**, **concave down**, and **inflection point**.



DF **Figure 1.39** Direction of bending: The graph is bent downward before $x = 0$ and bent upward after.

These definitions are only intuitive. Precise definitions can be given using basic geometry or ideas from calculus. A straight-line graph is not bent, so it is neither concave up nor concave down.

For example, the graph in Figure 1.39 is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. The direction of bending changes at the origin, so $(0, 0)$ is a point of inflection.

Rates of change can help us understand concavity. In **Figure 1.40**, we show the graph of a function that is increasing and concave up. Note that the upward concavity

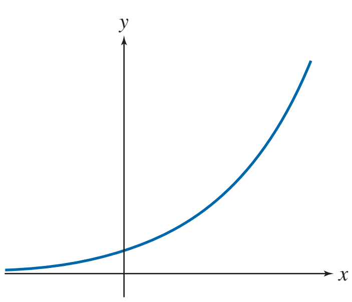


Figure 1.40 Graph of a function: This graph represents a function that is increasing and concave up.

causes the graph to get steeper as we move from left to right. Remember that steeper graphs indicate larger rates of change. We conclude that the function is increasing at an increasing rate. This type of observation is useful for any function and is of particular importance in applications.

EXAMPLE 1.19 Concavity and Rates of Change

Graphs are shown in Figure 1.41 and Figure 1.42. For each graph, determine which of the following statements applies.

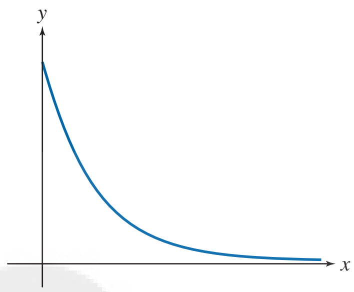


Figure 1.41 Graph of a function: This graph represents a function that is decreasing and concave up.

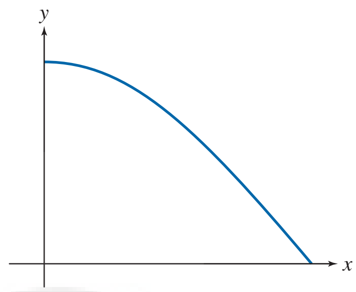


Figure 1.42 Graph of a function: This graph represents a function that is decreasing and concave down.

a. The corresponding function is decreasing at an increasing rate (that is, the rate of decrease is increasing).

b. The corresponding function is decreasing at a decreasing rate (that is, the rate of decrease is decreasing).

c. The corresponding function is increasing at an increasing rate.

d. The corresponding function is increasing at a decreasing rate.

SOLUTION

Figure 1.41: The graph flattens out as we move to the right. Thus, the function decreases rapidly at first but more slowly as we move to the right. The function is decreasing at a decreasing rate. Thus, statement b applies to this graph.

Figure 1.42: The graph gets steeper as we move to the right. Thus, the function decreases more rapidly as we move to the right. The function is decreasing at an increasing rate. We conclude that statement a applies to this graph.

TRY IT YOURSELF 1.19 Brief answers provided at the end of the section.

A graph that is increasing and concave down is shown in Figure 1.43. Which of the preceding statements a through d applies to this graph?

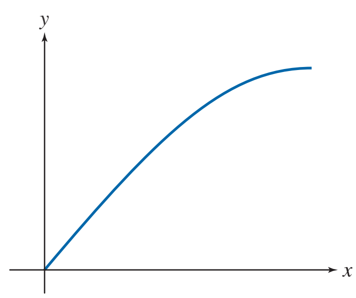


Figure 1.43 Graph of a function: This graph represents a function that is increasing and concave down.

EXTEND YOUR REACH

a. Can you make the graph of a function that is concave down on $(-\infty, \infty)$ yet has no maximum and no minimum?

b. Explain in terms of rates of change why a graph that is a straight line is neither concave up nor concave down.

EXAMPLE 1.20 Sketching Graphs with Given Concavity

Draw the graph of a function $y = f(x)$ that satisfies all of the following conditions, and identify the inflection points:

a. The domain is $[0, 10]$.

b. The function is increasing and concave up on $[0, 3)$.

c. The function is increasing and concave down on $(3, 5)$.

d. The function is decreasing and concave down on $(5, 7)$.

e. The function is decreasing and concave up on $(7, 10]$.

SOLUTION

The given conditions do not completely determine the graph, so there is no single correct answer. But the conditions do determine the general shape of the graph, and we show how each feature contributes to the construction of the graph.

a. The domain is $[0, 10]$, and that determines the horizontal extent of the graph.

b. We begin at $x = 0$ with an increasing function. Because it is concave up, the graph should get steeper as we move toward $x = 3$. This piece of the graph is shown in **Figure 1.44**.

c. Between $x = 3$ and $x = 5$, the graph is increasing and concave down. That makes the graph flatten out. In **Figure 1.45**, we have added this feature to the graph.

d. The function starts to decrease after $x = 5$. The downward concavity makes the graph steeper until we get to $x = 7$. This fact is shown in **Figure 1.46**.

e. From $x = 7$ on, the function continues to decrease, but because the graph is concave up, it begins to level off. The completed graph is shown in **Figure 1.47**.

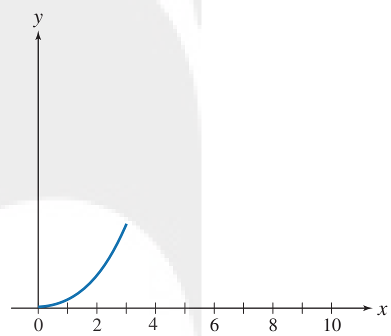


Figure 1.44 On the interval $[0, 3)$: The graph is increasing and concave up.

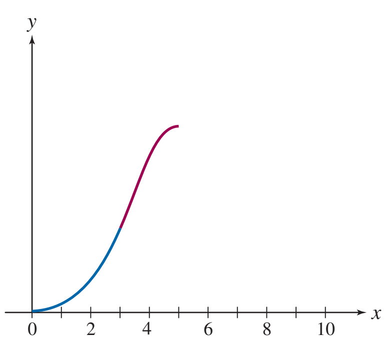


Figure 1.45 On the interval $(3, 5)$: The graph is increasing and concave down.

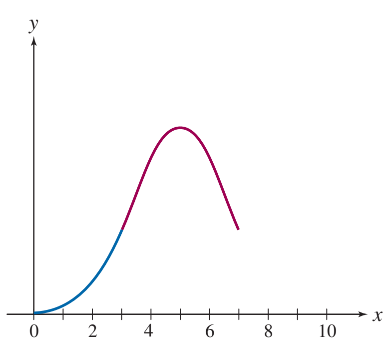


Figure 1.46 On the interval $(5, 7)$: The graph is decreasing and concave down.

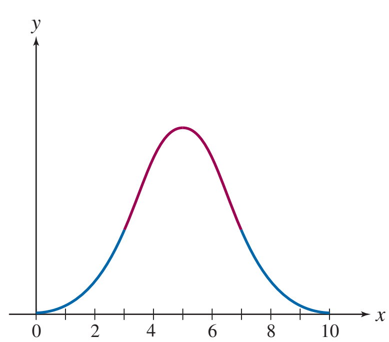


Figure 1.47 On the interval $(7, 10]$: The graph is decreasing and concave up.

The inflection points occur where the concavity changes. There are two such points, $(3, f(3))$ and $(7, f(7))$.

TRY IT YOURSELF 1.20 Brief answers provided at the end of the section.

Sketch the graph of a function that has a positive rate of change for $x < 0$, has a negative rate of change for $x > 0$, and is always concave down.

MODELS AND APPLICATIONS Sales Charts

Business models often include charts or graphs. The following example shows how mathematics applies to interpreting such figures.

EXAMPLE 1.21 Interpreting the Shape of Graphs: Sales

The graph in **Figure 1.48** shows the daily sales S in dollars for a certain company over a 1-year period. The horizontal axis shows the time t in months since the beginning of the year.

a. When did sales reach a (local) minimum?

b. Over what period were sales increasing?

c. Compare the growth in sales over the 4-month period from $t = 4$ to $t = 8$ with the growth in sales over the 4-month period from $t = 8$ to $t = 12$. Interpret your answer in terms of the shape of the graph.

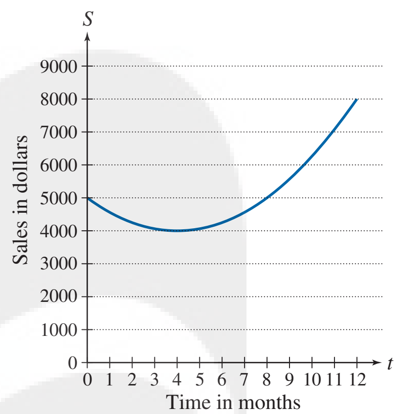


Figure 1.48 Sales as a function of time

SOLUTION

a. Sales reach a minimum where the graph is at a valley. That occurred at $t = 4$ (the beginning of May).

b. Sales are increasing when the graph is rising. That is from $t = 4$ to $t = 12$.

c. From $t = 4$ to $t = 8$, sales grew from \$4000 to \$5000, which is a growth of \$1000. From $t = 8$ to $t = 12$, sales grew from \$5000 to \$8000, which is a growth of \$3000. Sales grew faster over the later 4-month period. We expect this result because the graph gets steeper as we move to the right from $t = 4$: the graph is increasing and concave up.

TRY IT YOURSELF 1.21 Brief answers provided at the end of the section.

The graph in **Figure 1.49** shows the population of an endangered species over time. Over what period was the rate of decline in population slowing?

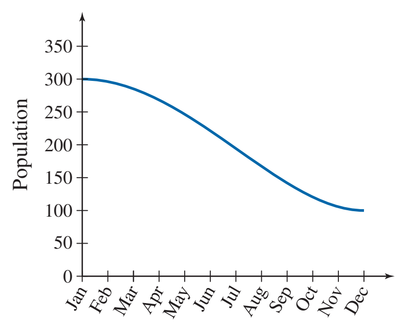
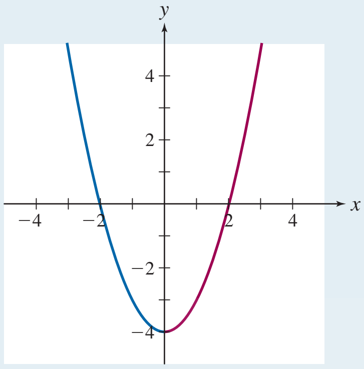


Figure 1.49 Population of an endangered species

TRY IT YOURSELF ANSWERS

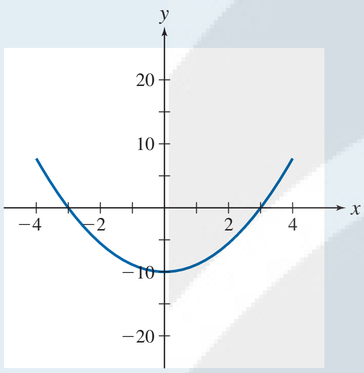
1.15 The rate of change is 4.

1.16



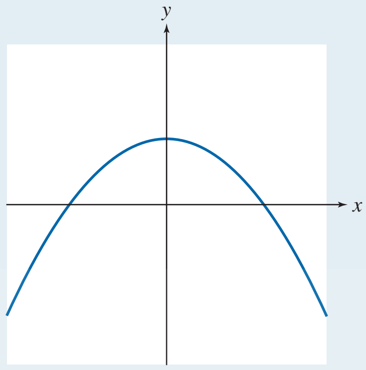
1.17 The function g decreases on $[-3, 2]$ and increases on $[2, 3]$. It has a local minimum value of $y = -2$ at $x = 2$. This is also an absolute minimum. There is no local maximum. The function has an absolute maximum value of $y = 4$, which occurs at $x = -3$.

1.18



1.19 Statement d applies: the function is increasing at a decreasing rate.

1.20



1.21 The rate of decline is slowing where the function is decreasing and concave up. That is from July through December.

EXERCISE SET 1.3

CHECK YOUR UNDERSTANDING

1. **True or false:** A local maximum may also be an absolute maximum.
2. What feature of a graph is measured by concavity?
3. If f is increasing and its graph is concave down, then as we move to the right, which is true?
   - **a.** The graph gets steeper.
   - **b.** The graph falls.
   - **c.** The graph becomes less steep.
   - **d.** None of the above.
4. **True or false:** The absolute minimum of a graph may occur at an endpoint.
5. A graph may have:
   - **a.** more than one local maximum
   - **b.** no local maximum
   - **c.** more than one local minimum
   - **d.** any of the above
6. What can be said about the rate of change of an increasing function on an interval?
7. A graph that has the shape of a frown is:
   - **a.** concave up
   - **b.** concave down
   - **c.** increasing
   - **d.** decreasing

8. Let f be a decreasing function with domain $[0, 2]$. Identify the location of the absolute minimum and the absolute maximum of f.

9. **True or false:** A decreasing function cannot have a local minimum.

10. **True or false:** If the graph of an increasing function is concave down, then the function is increasing at a decreasing rate.

11. The rate of change at a local maximum or local minimum is typically ____________.

SKILL BUILDING

12. **Maximum?** A friend says that $x = -1$ is a local maximum value of the function $f(x)$ whose graph is shown in **Figure 1.50**. Is she correct?

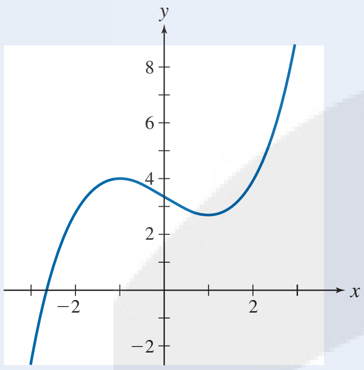


Figure 1.50 Possible local maximum value

Coloring graphs. In Exercises 13 through 23, a graph of a function f is given. Color the graphs as directed. You may trace the graph to color it.

13. Color green the portion of the graph in **Figure 1.51** where f is decreasing.

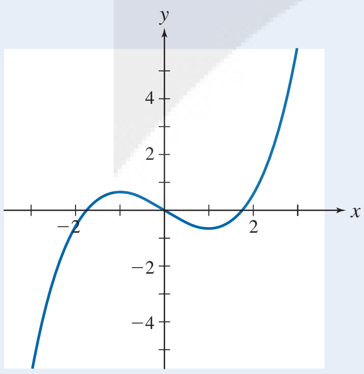


Figure 1.51 Graph for Exercises 13 through 15

14. Color green the portion of the graph in Figure 1.51 where f is concave up.

15. Color green the portion of the graph in Figure 1.51 that is between a local maximum and a local minimum.

16. Color green the portion of the graph in **Figure 1.52** where the rate of change of f is negative.

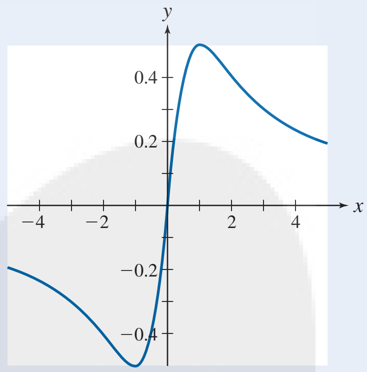


Figure 1.52 Graph for Exercises 16 through 18

17. Color green the portion of the graph in Figure 1.52 where the rate of change of f is positive.

18. Color green the portion of the graph in Figure 1.52 where the graph is concave up.

19. Color green the portion of the graph in **Figure 1.53** where f is increasing.

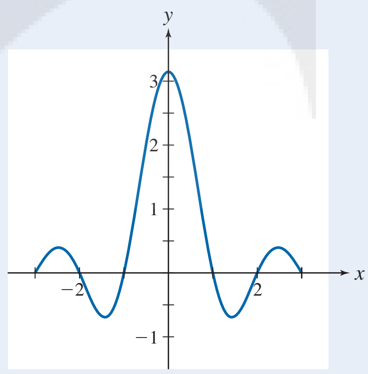


Figure 1.53 Graph for Exercises 19 through 21

20. Color green the region of the graph in Figure 1.53 containing the absolute maximum where the graph is concave down.

21. Color green the portion of the graph in Figure 1.53 where the graph is concave up.

22. Color green the portion of the graph in **Figure 1.54** where the function is increasing at a decreasing rate.

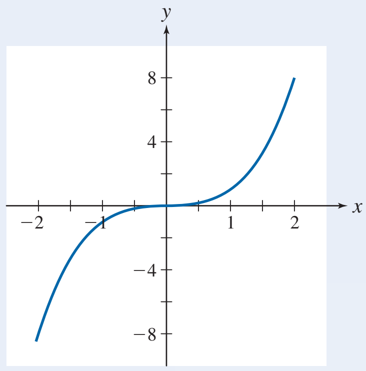


Figure 1.54 Graph for Exercises 22 and 23

23. Color green the portion of the graph in Figure 1.54 where the function is increasing at an increasing rate.

Features of graphs. The graph of a function $y = f(x)$ is shown with each of Exercises 24 through 32. You may assume that the domain of the function is indicated by the extent of the horizontal axis. Determine, using the given graph, each of the following:

a. Where the function is increasing, if anywhere

b. Where the function is decreasing, if anywhere

c. Where the graph is concave up, if anywhere

d. Where the graph is concave down, if anywhere

e. Where the graph has a point of inflection, if any

f. Where the function reaches a local maximum, if any

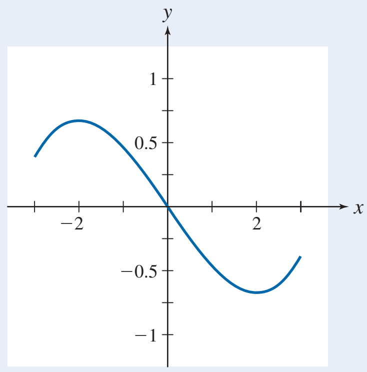
g. Where the function reaches a local minimum, if any

h. Where the function reaches an absolute maximum, if any

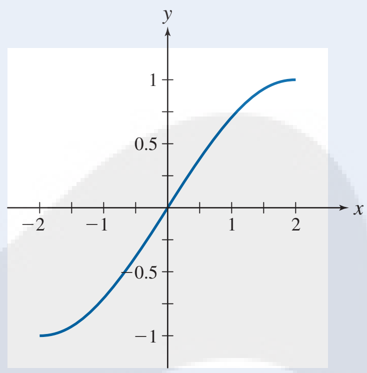
i. Where the function reaches an absolute minimum, if any

Remember that absolute maxima and minima may occur at endpoints of a graph.

24.



25.



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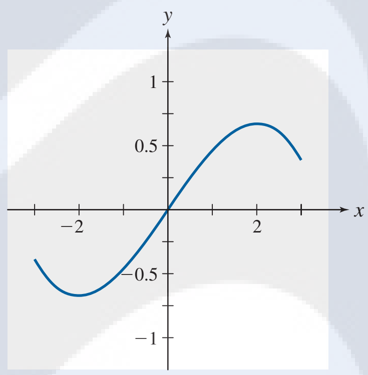
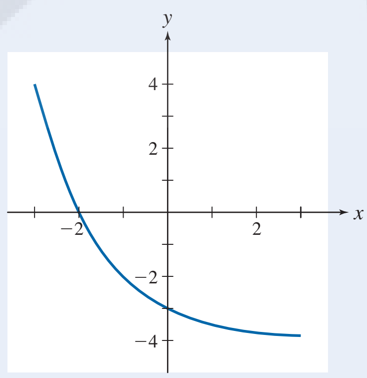
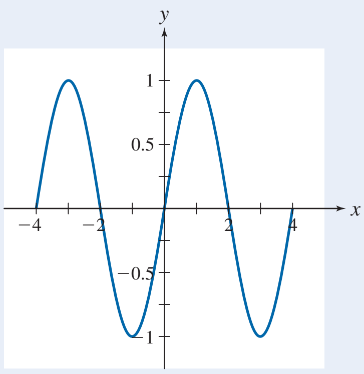


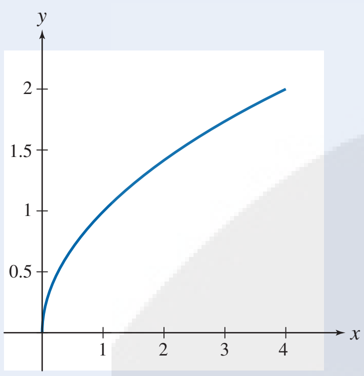
27.



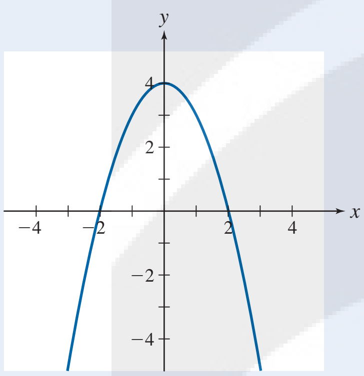
28.



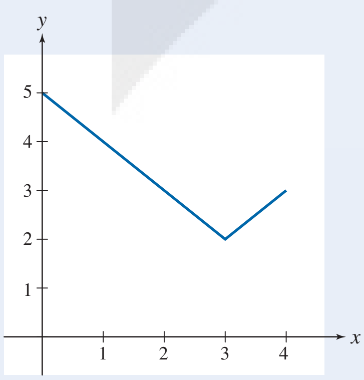
29.



30.

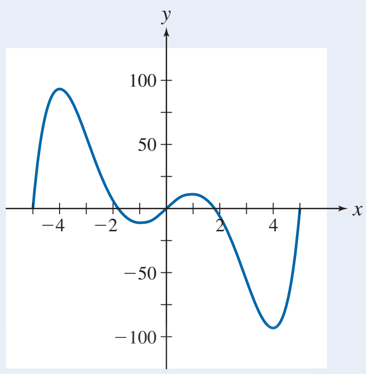


31.



For this graph, you will need to *estimate* the intervals of concavity and the location of the inflection points.

32.



Identifying rates of change. Exercises 33 through 38 refer to the graph of a function $y = f(x)$. For each function, estimate where the rate of change is:

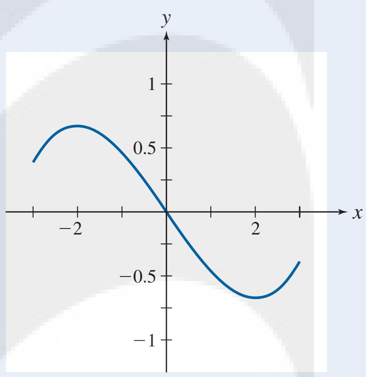
a. positive

b. negative

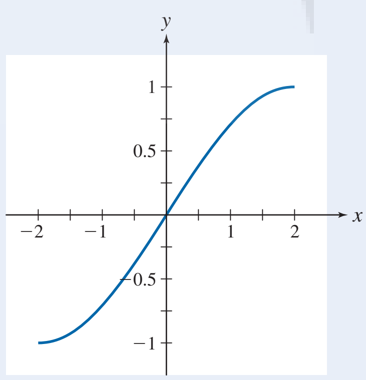
c. 0

In each case, you may assume the domain of the function is indicated by the extent of the x-axis. (Do not include the endpoints of the domain in your answers.)

33.



34.



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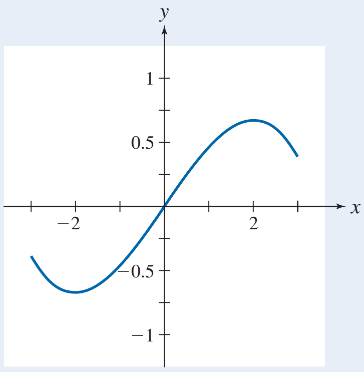
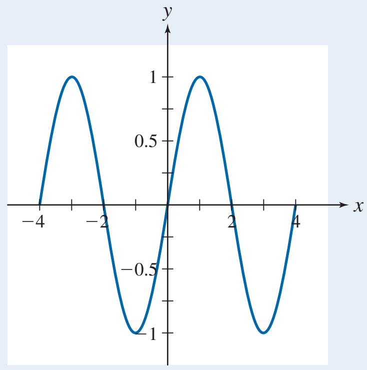


37.



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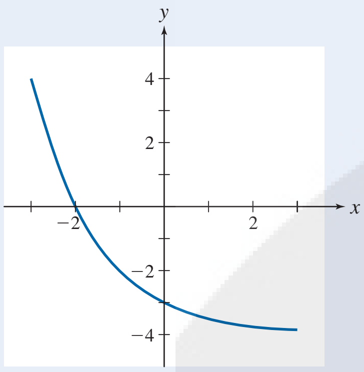
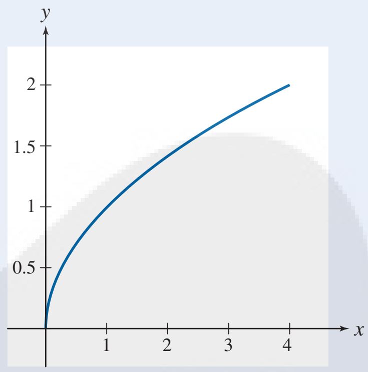


38.



PROBLEMS

Finding rates of change. In Exercises 39 through 42, you will calculate rates of change for functions.

39. The tangent line at $x = 1$ to the graph of $f(x) = x^3$ passes through the points $(1, f(1))$ and $(2, 4)$. Find the rate of change of f at $x = 1$.

40. The tangent line at $x = 2$ to the graph of $f(x) = x^3$ passes through the points $(2, f(2))$ and $(3, 20)$. Find the rate of change of f at $x = 2$.

41. The tangent line at $x = 1$ to the graph of $f(x) = \sqrt{x}$ passes through the points $(1, f(1))$ and $(5, 3)$. Find the rate of change of f at $x = 1$.

42. The tangent line at $x = 2$ to the graph of $f(x) = x - x^2$ passes through the points $(2, f(2))$ and $(4, -8)$. Find the rate of change of f at $x = 2$.

Sketching graphs. In Exercises 43 through 53, sketch a graph of a function with the given properties. There should be one graph for each exercise. For each exercise, there is more than one correct answer. Answers provided may serve as examples.

43. Increasing from $x = 0$ to $x = 2$, and decreasing after that

44. Local minimum at $x = 1$ and at $x = 3$, local maximum at $x = 2$

45. Point of inflection at $(2, 1)$

46. The rate of change is negative from $x = 0$ to $x = 2$ and from $x = 3$ to $x = 4$. The rate of change is positive elsewhere (except at $x = 0$, $x = 2$, $x = 3$, and $x = 4$).

47. Concave down to $x = 2$, concave up after that

48. Decreasing at an increasing rate from $x = 0$ to $x = 1$, decreasing and concave up from $x = 1$ to $x = 2$, and increasing and concave up from $x = 2$ to $x = 3$

49. Always decreasing, but at a decreasing rate

50. Concavity changes three times

51. The domain is $[-4, 4]$. There are an absolute maximum at $x = -4$ and an absolute minimum at $x = 4$. There are no local maxima or minima.

52. There are many (perhaps infinitely many) local maxima and local minima.

53. The domain of f is $[-2, 2]$ excluding the point $x = 0$. Also, f decreases on $[-2, 0)$ and on $(0, 2]$, but $f(-1) < f(1)$. *Suggestion*: Think in terms of a piecewise-defined function.

Familiar graphs. Exercises 54 through 63 involve functions whose graphs can be constructed easily by hand. The domain is assumed to be the set of all real numbers for which the given formula is defined.

54. Find the local and absolute maxima and minima of $f(x) = x^2$.

55. For $f(x) = x^2$, determine the regions where the rate of change is positive and where the rate of change is negative.

56. For $f(x) = x^2$, determine where the rate of change is zero.

57. Does the graph of $y = x^2$ have any points of inflection?

58. Let $y = 3x + 1$. Where is this function increasing, and where is it decreasing? Find all maxima and minima.

59. Find all points where the rate of change of $f(x) = x - 2$ is positive.

60. Let $y = 1 - x^2$. Find all local and absolute maxima and minima. Where is the graph concave up, and where is it concave down?

61. Let $y = x^3$. Locate all points of inflection.

62. For $f(x) = x^3$, are there any points where the rate of change is negative?

63. Let $y = \frac{1}{x}$. Determine the concavity.

64. Is there a maximum? Restrict the domain of $f(x) = \frac{1}{x}$ to the interval $(0, 1)$. Does f have an absolute maximum? Does it have an absolute minimum?

Finding maxima and minima. In Exercises 65 through 71, use a graphing utility to make the graph. Then give (approximately) any local maxima and minima as well as absolute maxima and minima. The answers that are provided are accurate to two decimal places. The accuracy of your answers will depend on the specific technology used and the preference of your instructor.

65. $f(x) = \dfrac{2^x}{1 + x^2}$. The domain is $[-2, 4]$.

66. $f(x) = 2x^3 + 3x^2 - 12x + 1$. The domain is $[-3, 3]$.

67. $f(x) = \dfrac{1}{\sqrt{x^2 + x + 3}}$. The domain is $[-3, 3]$.

68. $f(x) = \dfrac{x^2}{1 + x^4}$. The domain is $[-3, 3]$.

69. $f(x) = 6\sqrt{x} - x$. The domain is $[0, 25]$.

70. $f(x) = 4\sqrt{x} - x$. The domain is $[0, 25]$.

71. $f(x) = x^3 + x + 1$. The domain is $[-3, 3]$.

MODELS AND APPLICATIONS

72. A sunflower. Figure 1.55 shows the height in centimeters of a sunflower as a function of the time in days since germination. Over what (approximate) period is the graph concave up? What is happening to the growth rate of the flower over this period?

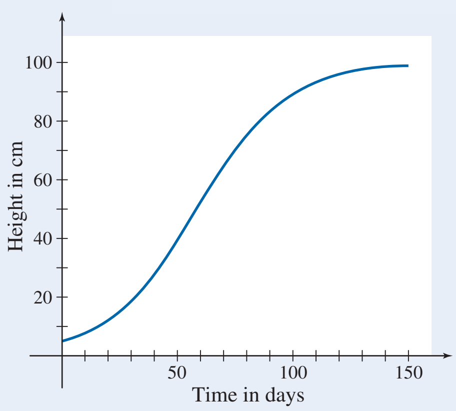


Figure 1.55 Growth of a sunflower

73. Photosynthesis. Figure 1.56 shows the net carbon dioxide exchange rates for 5-week-old rice plants as a function of the amount of light (in thousands of foot-candles) at 40°, 60°, and 80°. Values above the horizontal axis indicate absorption by the plant, and values below the axis indicate expulsion of carbon dioxide.

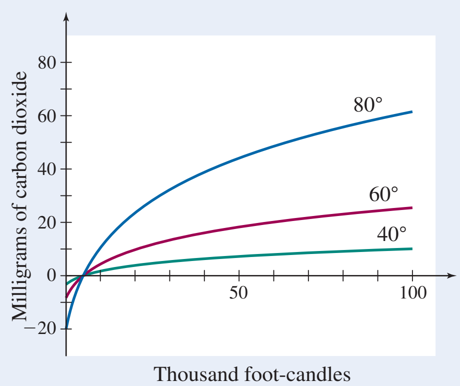


Figure 1.56 Carbon dioxide exchange

a. At which temperature do the rice plants show the least sensitivity to light?

b. At 80°, under which light condition would a change in light have a greater effect on the carbon dioxide exchange: dimmer light or brighter light?

74. **Going for ice cream.** The graph in Figure 1.57 shows your distance west of home, in yards, as a function of the time in minutes since you left home. You walk at a steady pace to the Dairy-berry store, where you enjoy an ice cream cone. You then start back home but stop to visit a friend.

a. How far from home is the Dairy-berry store?

b. How long did you spend eating ice cream?

c. How far is your friend's house from the ice cream store?

d. How is the "steady" pace of your walk to the store reflected in the shape of the graph over that period?

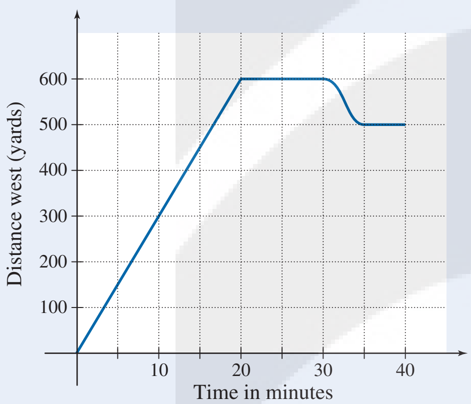


Figure 1.57 A trip for ice cream

75. **Windchill.** Wind speed affects perceived temperature, giving the effective temperature or temperature adjusted for windchill. The graph in Figure 1.58 shows the effective temperature as a function of the wind speed when the thermometer reads 20° Fahrenheit.

a. Does a higher wind speed cause a higher or a lower effective temperature?

b. Does a change in wind speed have a greater effect on effective temperature on a calm day or on a windy day?

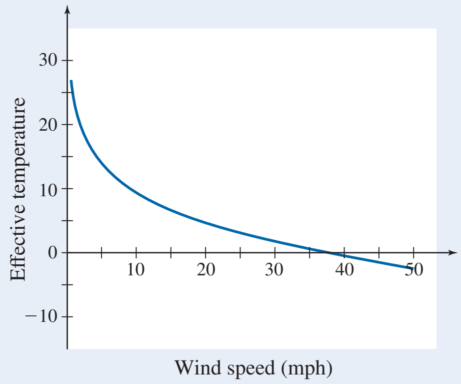


Figure 1.58 Windchill when the thermometer reads 20°

76. **A population of wolves.** The graph in Figure 1.59 shows the population N of wolves in a protected area t years after observation of the population began.

a. What (approximately) was the largest wolf population during the observation period, and when did that occur?

b. An extended drought severely limited the food supply available for wolves. Identify (approximately) the time period of the drought.

c. Was the wolf population growing faster early in the observation period or later?

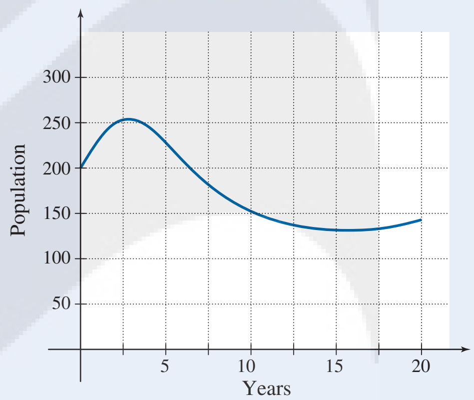


Figure 1.59 A population of wolves

77. **Alexander's formula.** Alexander's formula gives the (approximate) speed v, in meters per second, of a running animal in terms of the stride length s and hip height h, both measured in meters.[6] The relationship is

$$v = 0.78s^{1.67}h^{-1.17}$$

Fossilized dinosaur tracks are found, indicating a stride length of 3 meters.

[6] R. McNeill Alexander, "Estimates of Speeds of Dinosaurs," *Nature* 261: 129–130 (1976).

a. Plot the graph of speed versus hip height for hip heights up to 4 meters.

b. Suppose we estimate the hip height h and calculate the running speed. Does an error in hip height estimate lead to a greater error in the running speed calculation for dinosaurs of longer or shorter hip height?

78. Red shift. Many stellar objects are moving away from us at sufficient radial velocity to produce a significant red shift in the spectrum. The radial velocity v can be calculated from the red shift z using

$$v = c\left(\frac{(z+1)^2 - 1}{(z+1)^2 + 1}\right)$$

where c is the speed of light.

a. The red shift of the quasar 3C 273 is $z = 0.158$. (This means that the wavelengths of its spectral lines are stretched by 15.8%.) How fast is this quasar moving away from us? Report your answer as a percentage of the speed of light.

b. Use a graphing utility to make a graph of

$$y = \frac{(z+1)^2 - 1}{(z+1)^2 + 1}$$

c. Does the graph show an increasing function or a decreasing function? What is the concavity of the graph?

d. What does the information from part c reveal about the relationship between the red shift and the radial velocity?

79. Main-sequence stars. Main-sequence stars are common stars like our own sun. The Hertzsprung-Russell diagram, or H-R diagram, in **Figure 1.60** shows for main-sequence stars the relation between the surface temperature T, in kelvins, and the relative luminosity L, which is the ratio of a star's luminosity to that of the sun. Thus, the sun has relative luminosity 1, and a star that is four times as bright as the sun has a relative luminosity of 4. Be careful in using this graph: the horizonal axis is in reverse order, as is standard for the H-R diagram.

a. What is the surface temperature of the sun?

b. Is the relative luminosity for a main sequence star an increasing function of surface temperature or a decreasing function of surface temperature?

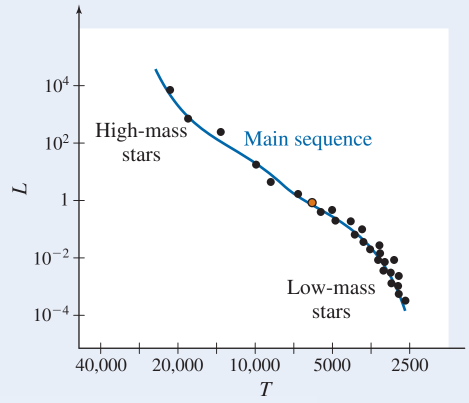


Figure 1.60 H-R diagram for main-sequence stars

80. Supply and demand curves. A supply curve is a graph that shows the quantity of a product that is made available by suppliers as a function of the price. Similarly, a demand curve is a graph that shows the quantity of a product consumers are willing to purchase as a function of the price. For example, the red graph in **Figure 1.61** is a supply curve, and the blue graph is a demand curve.

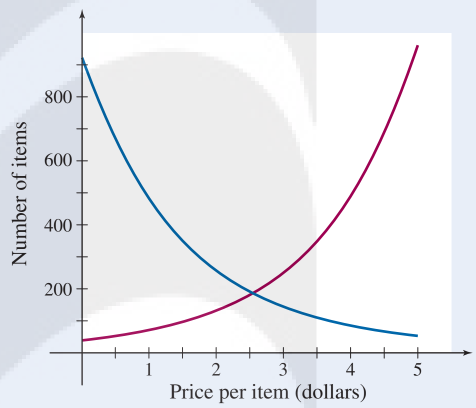


Figure 1.61 Supply (red) and demand (blue) curves

a. Is the supply curve in Figure 1.61 increasing or decreasing? Explain your answer in terms of the availability of items as the price increases.

b. Is the demand curve in Figure 1.61 increasing or decreasing? Explain your answer in terms of the quantity of items in demand as the price increases.

c. The equilibrium price is the price where demand matches supply. What is the equilibrium price in this case?

CHALLENGE EXERCISES FOR INDIVIDUALS OR GROUPS

81. Finding a rate of change. Use the fact that the graph of $f(x) = 1$ is a horizontal line to find the rate of change of f at any point. *Suggestion*: First draw the graph, and then sketch the tangent line.

82. A horse race. A racecourse runs due west of the starting gate. Let $R(t)$ denote the distance west of the starting gate of the roan horse t seconds after the beginning of the race, and let $G(t)$ denote the distance west of the starting gate for the gray mare. At some point in the race, the gray mare is ahead of the roan. From that time on, the rate of change of $G(t)$ is at least as large as the rate of change of $R(t)$. Bearing in mind that the rate of change of distance west is the velocity, which horse wins the race? Explain your reasoning.

83. A possible graph? Is it possible to have a function f with all of the following properties?

- The domain of f is $[0, \infty)$.
- f is increasing.
- $f(x) < 1$ for all x.

If not, explain why. If so, produce the required function and graph.

84. Another possible graph? Find a function g with the same properties as in Exercise 83 except that the domain is $(-\infty, \infty)$. *Suggestion*: Think in terms of a piecewise-defined graph.

85. Calculating a rate of change. Recall that the rate of change of a function f at x is defined in terms of the average rate of change of f on shorter and shorter intervals with x as an endpoint. In this exercise, we use this definition to calculate the rate of change of $f(x) = x^2$ at the point $x = 1$.

a. Show that the average rate of change of $f(x) = x^2$ on the interval $[1, 1 + h]$ equals $2 + h$. (Assume that $h > 0$.)

b. Describe what happens to the average rate of change from part a as the length of the interval gets very small—that is, when h is near 0. What does your answer give for the rate of change of $f(x) = x^2$ at the point $x = 1$?

c. Use your answer to part b to find the tangent line to the graph $y = x^2$ corresponding to the point $x = 1$. Then make a careful sketch of the graph and the tangent line.

REVIEW AND REFRESH: Exercises from Previous Sections

86. From Section P.1: Find the distance between the points $(1, 4)$ and $(2, 5)$.

Answer: $\sqrt{2}$

87. From Section P.2: Solve the inequality $|x - 4| < 3$.

Answer: $(1, 7)$

88. From Section P.3: Solve the inequality $x(x + 1)(x - 3) > 0$.

Answer: $(-1, 0) \cup (3, \infty)$

89. From Section 1.1: If $f(x) = x^2$, simplify $\dfrac{f(x + h) - f(x)}{h}$ for $h \neq 0$.

Answer: $2x + h$

90. From Section 1.1: Find the domain of $\dfrac{\sqrt{x}}{x^2 - x - 2}$.

Answer: $[0, 2) \cup (2, \infty)$

91. From Section 1.2: If $f(x) = \dfrac{1}{x^2}$, find the average rate of change of $f(x)$ from $x = 1$ to $x = 2$.

Answer: $-\dfrac{3}{4}$

92. From Section 1.2: If $f(x) = x^2 - 3$, calculate the average rate of change of $f(x)$ on the interval $[x, x + h]$. Assume that $h > 0$, and simplify your answer.

Answer: $2x + h$

1.4 Limits and End Behavior of Graphs

1.1 The Basics of Functions
1.2 Average Rates of Change
1.3 Graphs and Rates of Change
1.4 Limits and End Behavior of Graphs

The limit at infinity describes the long-term behavior of functions.

In this section, you will learn to:

1. Explain the concept of the long-term behavior, or limit, of a function.
2. Use the appropriate notation to represent the limit of a function.
3. Calculate elementary limits.
4. Estimate the limit of a function using a graph or a table.
5. Interpret the limit of a function in applications.

When a skydiver (**Figure 1.62**) jumps from an airplane, gravity causes her downward velocity to increase rapidly. But air resistance works against the downward pull of gravity, and the faster she goes, the greater the opposing force due to air resistance. Eventually, air resistance almost matches the pull of gravity, and the skydiver's downward velocity levels out at a terminal velocity of about 120 miles per hour. This is an example of how the limiting behavior of processes is important in life.

Figure 1.62 A skydiver Brian Erler/Getty Images

Limiting values are also important in medicine. When your doctor gives you an injection of an antibiotic, the drug level in your bloodstream increases rapidly. But after a time, it reaches a maximum, and then it slowly decays. Eventually, the drug level approaches the normal, limiting value of zero.

Consider also the study of population growth. With plentiful resources, an animal or human population may grow rapidly. But as the population increases, there is greater and greater competition for resources that are finite. The result is that population growth eventually slows, and population levels stabilize.

An Intuitive Discussion of Limits at Infinity

The limit at infinity is understood in terms of the end behavior of the graph.

Let's look more closely at the skydiver. Her downward velocity t seconds after she has jumped from the airplane, and before the parachute opens, is given by

$$V(t) = 120(1 - 0.75^t) \text{ miles per hour}$$

The accompanying table of values and the graph in **Figure 1.63** help us understand what the formula means to the skydiver.

Time t	Velocity $V(t)$
0	0
5	91.52
10	113.24
15	118.40
20	119.62
25	119.91

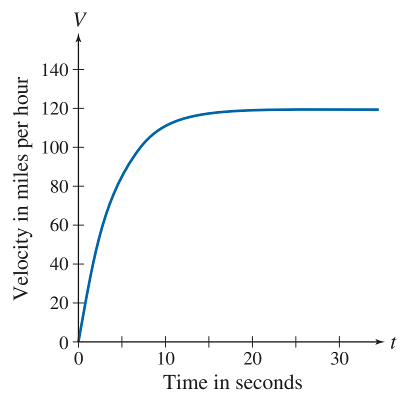


Figure 1.63 The velocity of a skydiver

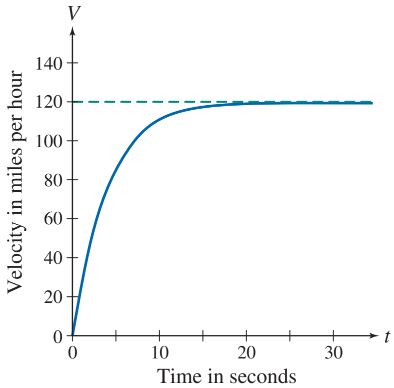


Figure 1.64 Limiting value: The skydiver's velocity approaches 120 miles per hour.

Both the table of values and the graph indicate that, as time passes, the skydiver's velocity stabilizes as it approaches 120 miles per hour. To further illustrate the point, in **Figure 1.64** we have added the horizontal line $y = 120$ to the graph of $V(t)$. This line clearly shows the long-term behavior or end behavior of the function $V(t)$ and its graph.

We use the notation $V(t) \to 120$ as $t \to \infty$ to indicate that, for large values of t, $V(t)$ approaches the value 120. Many calculus texts use the notation $\lim_{t\to\infty} V(t) = 120$. Both indicate the limiting value of $V(t)$ or the limit as t goes to infinity of $V(t)$.

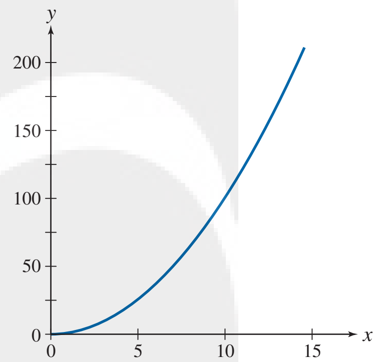
Examples of Limits at Infinity

Some common patterns emerge in the end behavior of graphs.

A number of different things may happen at the tail ends of graphs. A few examples of long-term behavior are illustrated by the graphs of $y = x^2$, $y = -x^2$, and $y = \dfrac{1}{x^2}$.

Long-term behavior of $y = x^2$: When x gets larger and larger, x^2 gets larger too. This fact is illustrated by both the accompanying table of values and the graph in **Figure 1.65**, which rises without any bound as we move to the right.

x	$y = x^2$
10	100
100	10,000
1000	1,000,000
100,000	10,000,000,000



DF **Figure 1.65** $x^2 \to \infty$ as $x \to \infty$: The function $y = x^2$ increases without bound as x increases.

We indicate this fact by writing

$$x^2 \to \infty \text{ as } x \to \infty$$

This notation is read "x^2 approaches infinity as x approaches infinity." Many calculus texts use the notation

$$\lim_{x\to\infty} f(x) = \infty$$

Long-term behavior of $y = -x^2$: When x gets larger and larger, $-x^2$ gets larger in magnitude too, but with a minus sign. Thus, the function decreases without bound. This fact is illustrated by the accompanying table of values and the graph in **Figure 1.66**, which falls indefinitely as we move to the right.

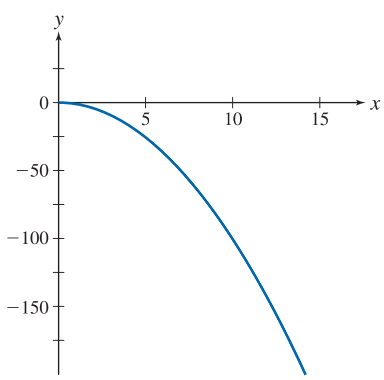


Figure 1.66 $-x^2 \to -\infty$ as $x \to \infty$: The function $y = -x^2$ decreases without bound as x increases.

x	$y = -x^2$
10	−100
100	−10,000
1000	−1,000,000
100,000	−10,000,000,000

We indicate this fact by writing

$$-x^2 \to -\infty \text{ as } x \to \infty$$

This notation is read "$-x^2$ approaches negative infinity as x approaches infinity."

Long-term behavior of $y = \frac{1}{x^2}$: A large denominator (in relation to the size of the numerator) means a small fraction. For example, 10,000 is a very large number, so $1/10{,}000 = 0.0001$ is a very small number. This example suggests that when x is large, $y = \frac{1}{x^2}$ is very small. That is, as x gets larger and larger, $\frac{1}{x^2}$ gets closer and closer to zero. In symbols, $\frac{1}{x^2} \to 0$ as $x \to \infty$. This notation is read "$\frac{1}{x^2}$ approaches zero as x approaches infinity." In calculus texts, this fact is often written as $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

The fact that the limit is 0 means the graph shown in **Figure 1.67** approaches the x-axis at its right tail end. The behavior of $\frac{1}{x^2}$ for increasing x can also be seen in the accompanying table of values.

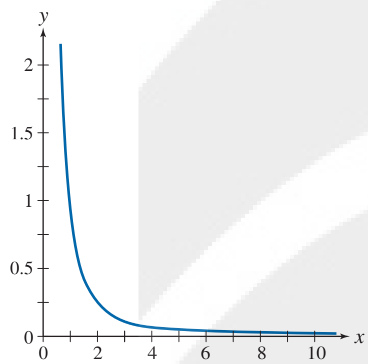


Figure 1.67 $\frac{1}{x^2} \to 0$ as $x \to \infty$: The graph of $\frac{1}{x^2}$ gets close to the x-axis as x increases.

x	$\frac{1}{x^2}$
10	0.01
100	0.0001
1000	0.000001
100,000	0.0000000001

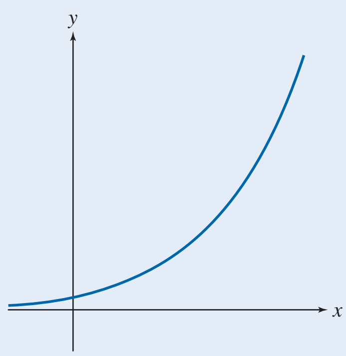
In our discussion, we have considered the right tail end of the graph of f in connection with the long-term behavior of $f(x)$. In a similar way, we can study the left tail end of the graph of f. In this situation, we ask about the behavior of f when x is large in size but negative, and we write $x \to -\infty$.

CONCEPTS TO REMEMBER: Intuitive Notion of Limits

- The notation $f(x) \to \infty$ as $x \to \infty$ is read "$f(x)$ approaches infinity as x approaches infinity." It means that $f(x)$ grows without bound as x gets very large, and that the graph rises indefinitely at the right tail end.

  Example:

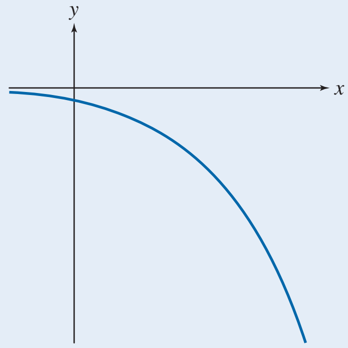
  For $n > 0$, $x^n \to \infty$ as $x \to \infty$



- The notation $f(x) \to -\infty$ as $x \to \infty$ is read "$f(x)$ approaches negative infinity as x approaches infinity." It means that $f(x)$ decreases without bound as x gets very large, and that the graph falls indefinitely at the right tail end.

  Example:

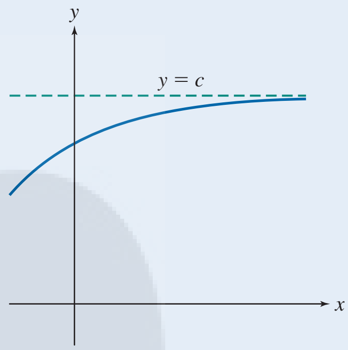
  For $n > 0$, $-x^n \to -\infty$ as $x \to \infty$



- The notation $f(x) \to c$ as $x \to \infty$ is read "$f(x)$ approaches c as x approaches infinity." It means that $f(x)$ gets arbitrarily close to c as x gets very large and that the graph approaches the line $y = c$ at the right tail end.

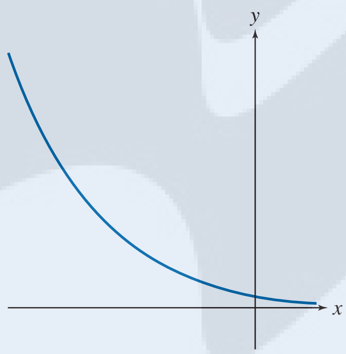
  Example:

  For $n > 0$, $\dfrac{1}{x^n} \to 0$ as $x \to \infty$

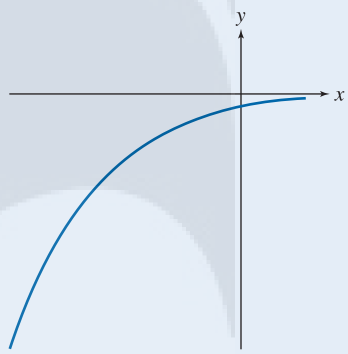


- This reasoning also applies to the case when x is negative and x gets very large in size. For this, we replace $x \to \infty$ by $x \to -\infty$, and the results describe the left tail end of the graph rather than the right.

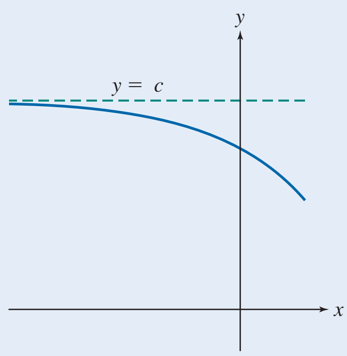
$f(x) \to \infty$ as $x \to -\infty$



$f(x) \to -\infty$ as $x \to -\infty$



$f(x) \to c$ as $x \to -\infty$



When we calculate limits involving fractions, it is worth emphasizing that the size of a fraction depends on the relative sizes of the numerator and denominator. Just knowing, for example, that the denominator is large doesn't automatically make the fraction small, because the numerator may be much larger than the denominator. Also, not all functions have a limit at infinity.

Elementary Limit Calculations

The intuitive notion of limits is used to perform simple limit calculations.

In the following examples, we use the results just discussed to find limits at infinity. In each example, we first explain how to find the limiting value and then support our explanation with both a table of values and a graph.

EXAMPLE 1.22 A Function That Decreases Without Bound

Find the limit as $x \to \infty$ of $y = -x^3$.

SOLUTION

We know that as x increases x^3 grows without bound. Hence, $-x^3$ grows without bound in the negative direction. We conclude that

$$-x^3 \to -\infty \text{ as } x \to \infty$$

The limit calculation is reinforced by the accompanying table. This means the graph in Figure 1.68 falls indefinitely.

x	$-x^3$
1	−1
10	−1000
100	−1,000,000
10,000	−1,000,000,000,000

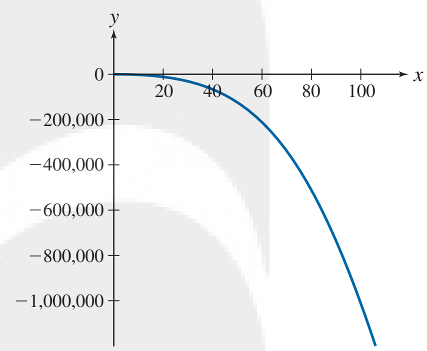


Figure 1.68 $-x^3 \to -\infty$ as $x \to \infty$

TRY IT YOURSELF 1.22 Brief answers provided at the end of the section.

Calculate the limit as $x \to -\infty$ of $y = -x^3$, and show the left-hand tail of the graph.

EXTEND YOUR REACH

Earlier in this section we gave three examples to illustrate typical behavior of limits as $x \to \infty$. Find corresponding examples to illustrate typical behavior of limits as $x \to -\infty$. The limit in Try It Yourself 1.22 shows one.

EXAMPLE 1.23 A Graph That Approaches the *x*-Axis

Find the limit as $x \to \infty$ of $y = \dfrac{100}{x+1}$.

SOLUTION

When x is very large, in particular much larger than 100, the denominator $x + 1$ is much larger than the numerator 100. Hence, the fraction $\dfrac{100}{x+1}$ is near 0 when x is large. We conclude that

$$\frac{100}{x+1} \to 0 \text{ as } x \to \infty$$

The accompanying table of values supports our assertion that the limit is 0. As a consequence, the graph in **Figure 1.69** approaches the x-axis when x approaches its right tail end.

x	$\dfrac{100}{x+1}$
10	9.0909
100	0.9901
1000	0.0999
10,000	0.0100
100,000	0.0010

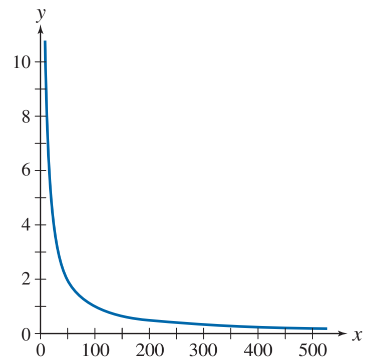


Figure 1.69 $\dfrac{100}{x+1} \to 0$ as $x \to \infty$

TRY IT YOURSELF 1.23 Brief answers provided at the end of the section.

Calculate the limit as $x \to -\infty$ of $y = \dfrac{5}{x}$, and show the left-hand tail of the graph.

EXTEND YOUR REACH What do you think is the limit as $x \to \infty$ of the rate of change of the function $y = \dfrac{100}{x+1}$? *Suggestion*: Use a straightedge with the graph in Figure 1.69 to estimate the slopes of tangent lines as we move to the right.

EXTEND YOUR REACH The following limits are difficult to calculate exactly. Use a calculator to make a table of values and a graph. Then use those to estimate the limits.

1. Estimate the limit as $x \to \infty$ of $y = \dfrac{200}{4 + 2^{-x}}$.
2. Estimate the limit as $x \to \infty$ of $y = x(2^{1/x} - 1)$.
3. Estimate the limit as $x \to \infty$ of $y = \dfrac{x^2}{2^x}$.

Breaking Limit Calculations into Simpler Steps

Computing limits of complicated functions is accomplished in steps.

The calculation of limits can often be broken down into simpler steps. Consider, for example, the limit as $x \to \infty$ of $y = \dfrac{1}{x^2} + 3$. We know that $\dfrac{1}{x^2} \to 0$ as $x \to \infty$. Then, because $\dfrac{1}{x^2}$ is near 0, the expression $\dfrac{1}{x^2} + 3$ is near 3. The following shorthand notation is both useful and common:

$$\text{As } x \to \infty,\ \overbrace{\cancel{\frac{1}{x^2}}}^{0} + 3 \to 3$$

Thus, $\dfrac{1}{x^2} + 3 \to 3$ as $x \to \infty$.

Both numerical and graphical evidence support our assertion that the limit is 3. Note that the right tail end of the graph shown in **Figure 1.70** approaches the line $y = 3$. The accompanying table of values also illustrates the end behavior.

x	$y = \frac{1}{x^2} + 3$
10	3.01
100	3.0001
1000	3.000001
100,000	3.0000000001

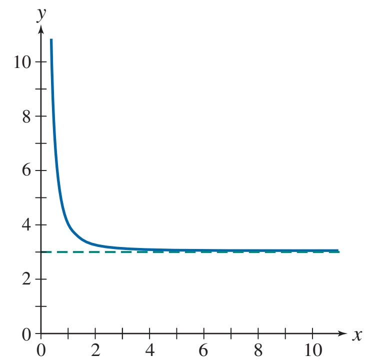


Figure 1.70 $\frac{1}{x^2} + 3 \to 3$ as $x \to \infty$: The graph of $y = \frac{1}{x^2} + 3$ approaches the line $y = 3$ as x increases.

EXAMPLE 1.24 Steps in Limit Calculations

a. Find the limit as $x \to \infty$ of $y = 5 + \frac{100}{x}$.

b. Observe that $\frac{1+12x}{1+3x} = \frac{\frac{1}{x}+12}{\frac{1}{x}+3}$ when $x \neq 0$. Use this fact to find the limit as $x \to \infty$ of $y = \frac{1+12x}{1+3x}$.

SOLUTION

a. When x is much larger than 100, the fraction $\frac{100}{x}$ is near 0. Hence,

$$\text{As } x \to \infty,\ 5 + \overbrace{\frac{100}{x}}^{0} \to 5$$

We conclude that

$$5 + \frac{100}{x} \to 5 \text{ as } x \to \infty$$

x	$5 + \frac{100}{x}$
10	15
100	6
1000	5.1
10,000	5.01
100,000	5.001

The accompanying table of values supports our calculation. This limit means that the graph in **Figure 1.71** approaches the line $y = 5$.

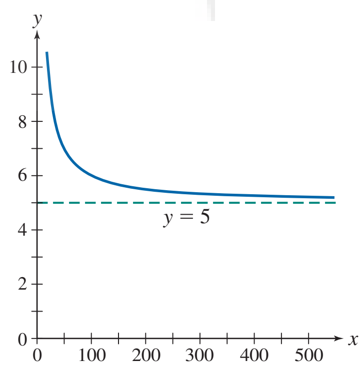


Figure 1.71 $5 + \frac{100}{x} \to 5$ as $x \to \infty$

b. We establish the equality by dividing both the top and the bottom of the fraction by x:

$$\frac{1+12x}{1+3x} = \frac{\left(\frac{1}{x}\right)(1+12x)}{\left(\frac{1}{x}\right)(1+3x)} \qquad \text{Divide top and bottom by } x \text{ for } x \neq 0.$$

$$= \frac{\frac{1}{x}+12}{\frac{1}{x}+3} \qquad \text{Simplify.}$$

We use this form of the fraction to find the limit:

$$\frac{\frac{1}{x}+12}{\frac{1}{x}+3} \to \frac{\overset{0}{\cancel{\frac{1}{x}}}+12}{\underset{}{\overset{0}{\cancel{\frac{1}{x}}}}+3} \text{ as } x \to \infty \qquad \text{Use } \frac{1}{x} \to 0 \text{ as } x \to \infty.$$

$$= \frac{12}{3} = 4 \qquad \text{Simplify.}$$

We conclude that $\frac{1+12x}{1+3x} \to 4$ as $x \to \infty$.

The accompanying table of values supports our calculation. The limit shows that the graph in **Figure 1.72** approaches the line $y = 4$.

x	$\frac{1+12x}{1+3x}$
10	3.903226
100	3.990033
1000	3.999000
10,000	3.999900

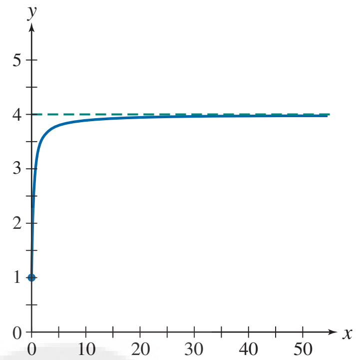


Figure 1.72 $\frac{1+12x}{1+3x} \to 4$ as $x \to \infty$

TRY IT YOURSELF 1.24 Brief answers provided at the end of the section.

a. Find the limit as $x \to \infty$ of $f(x) = \frac{2}{1+\frac{1}{x}}$.

b. Observe that $\frac{4x+1}{2x-1} = \frac{4+\frac{1}{x}}{2-\frac{1}{x}}$ when $x \neq 0$. Use this fact to find the limit as $x \to \infty$ of $y = \frac{4x+1}{2x-1}$.

EXTEND YOUR REACH

The approach used in Example 1.24 can be generalized to work for other limits. Use the given suggestions to calculate the following limits.

1. Find the limit as $x \to \infty$ of $y = \frac{2x^2+1}{x^2+3}$. *Suggestion*: Divide top and bottom by x^2.
2. Find the limit as $x \to \infty$ of $y = \frac{x^2+1}{x+1}$. *Suggestion*: Divide top and bottom by x.
3. Find the limit as $x \to \infty$ of $y = \frac{x+1}{x^2+1}$. *Suggestion*: Divide top and bottom by x^2.
4. Find the limit as $x \to \infty$ of $y = \frac{2x^2+ax+b}{x^2+cx+d}$. *Suggestion*: Divide top and bottom by x^2. Do the values of the constants a, b, c, and d affect the value of the limit?

Example 1.24 suggests the following strategy, which is often helpful in calculating limits of fractions. For a function of the form $y = \frac{ax+b}{cx+d}$, the calculations of limits at $\pm\infty$ are facilitated by dividing the top and the bottom of the fraction by x:

$$\frac{ax+b}{cx+d} = \frac{a+\frac{b}{x}}{c+\frac{d}{x}}, \qquad x \neq 0$$

The concept of a limit will prove to be invaluable in describing graphs of new functions in coming chapters. For example, in Chapter 3 we will show how to calculate limits of exponential functions; in Chapter 4 we will calculate limits of logarithmic functions; and in Chapter 5 we will calculate limits of polynomials and rational functions.

MODELS AND APPLICATIONS Long-Term Behavior

In applications, limits at infinity express what happens in the long run. For example, suppose a cup of tea is left on the counter. Let $T(t)$ denote the temperature, in degrees, of the tea t minutes after it is left there. Then the limit as $t \to \infty$ of $T(t)$ is the temperature of the tea after a long time. Eventually, the temperature matches that of the room, say 70°. In terms of limits, $T(t) \to 70$ as $t \to \infty$.

EXAMPLE 1.25 The Meaning of Limits in Applied Settings

a. Water containing 20% salt flows into a tank of initially pure water, and the well-mixed solution flows out at the same rate. Let $S(t)$ denote the percentage of salt in the tank t hours after the process begins. Explain in terms of the percentage of salt the meaning of the following expression: $S(t) \to 20\%$ as $t \to \infty$.

b. A population is growing in an environment that can support 500 individuals. If $N(t)$ is the population at time t, use limit notation to express the statement "The population will eventually grow toward 500 individuals."

SOLUTION

a. The limit notation tells us that $S(t)$ tends to a value of 20% in the long run. This means that, after a long time, the solution in the tank will approach 20% salt.

b. Because the population grows to 500, the function $N(t)$ approaches 500 for large values of t. In terms of limits,

$$N(t) \to 500 \text{ as } t \to \infty$$

TRY IT YOURSELF 1.25 Brief answers provided at the end of the section.

The spacecraft *Voyager 1* is currently about 14 billion miles from the sun and is moving away at a constant velocity. In space, there is nothing to slow the craft. If $D(t)$ is the distance from *Voyager 1* to the sun at time t, what is the limit as $t \to \infty$ of $D(t)$?

EXAMPLE 1.26 Functional Response

The number P of prey taken by a predator depends on the abundance x of the prey. The relationship proposed by C. S. Holling[7] is known as the functional response:

$$P(x) = \frac{ax}{1+bx}$$

Here a and b are constants that are determined by the specific predator and prey involved, and $b \neq 0$.

a. Find the limit as $x \to \infty$ of $P(x)$.

b. Explain the meaning of the limit you calculated in part a in terms of the number of prey taken by a predator.

SOLUTION

a. We divide the top and the bottom of the fraction by x in order to facilitate the limit calculation.

$$\frac{ax}{1+bx} = \frac{a}{\frac{1}{x}+b} \qquad \text{Divide top and bottom by } x,\ x \neq 0.$$

as $x \to \infty$

$$\frac{a}{\frac{1}{x}+b} \to \frac{a}{\overset{0}{\cancel{\frac{1}{x}}}+b} = \frac{a}{b} \qquad \text{Use } \frac{1}{x} \to 0 \text{ as } x \to \infty.$$

[7]C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist* 91: 385-398 (1959).

We conclude that

$$P(x) \to \frac{a}{b} \text{ as } x \to \infty$$

b. To say that x is large means that the abundance of prey is large. Thus, the limit $\frac{a}{b}$ is the largest number of prey the predator will take no matter how much prey is available.

TRY IT YOURSELF 1.26 Brief answers provided at the end of the section.

If a lake is stocked with N_0 fish of the same age, then the number C of fish caught by fishing over the life span of the fish is

$$C(x) = \frac{xN_0}{1+x}$$

Here x represents the ratio of fish caught annually to fish that die annually of natural causes. This is one form of the catch equation.

a. Find the limit as $x \to \infty$ of $C(x)$.

b. Explain the meaning of the limit in terms of the ratio of fish caught annually to fish dying annually.

In the next chapter, we will learn how to calculate limits of exponential functions. But for now, we will rely on graphs and tables of values to make estimates.

EXAMPLE 1.27 Population Growth

The number N of a protected species in a controlled environment after t years is given by

$$N = \frac{1000}{1 + 19 \times 0.7^t}$$

Because the resources of the controlled environment are limited, we expect the population N to tend toward a limiting value.

a. Make a table of values showing the population in years $t = 0, 5, 10, 15, 20, 25, 30$. Round your data to the nearest whole number.

b. Use a graphing utility to plot the graph of N versus t over the first 30 years.

c. Based on the table of values from part a and the graph from part b, estimate the value of the limit as $t \to \infty$ of N.

d. Explain the meaning of the limit you calculated in part c in terms of the population.

SOLUTION

a. We used a calculator to make the accompanying table of values.

t	N
0	50
5	238
10	651
15	917
20	985
25	997
30	1000

b. The graph is in **Figure 1.73**.

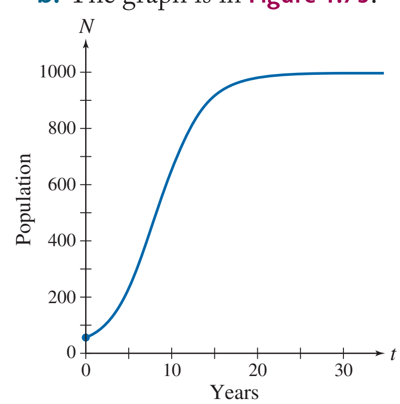


Figure 1.73 The population over a 30-year period

c. Both the graph and the table of values suggest a limiting value of around 1000.

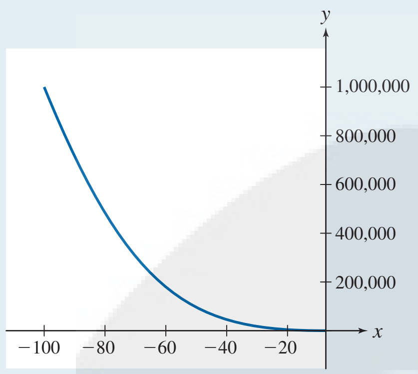
d. In the long term, we expect the population to grow and stabilize at around 1000 individuals.

TRY IT YOURSELF 1.27 Brief answers provided at the end of the section.

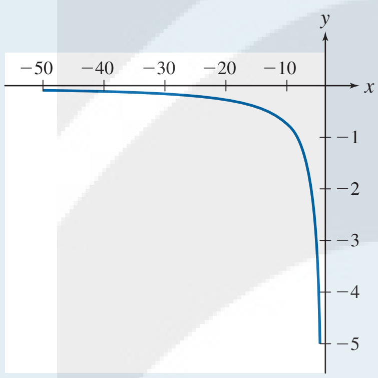
Repeat the example using a growth model of $N = \dfrac{1000}{1 - 0.5 \times 0.7^t}$.

TRY IT YOURSELF ANSWERS

1.22 $-x^3 \to \infty$ as $x \to -\infty$



1.23 $\dfrac{5}{x} \to 0$ as $x \to -\infty$



1.24 **a.** $\dfrac{2}{1+\frac{1}{x}} \to 2$ as $x \to \infty$

b. $\dfrac{4x+1}{2x-1} \to 2$ as $x \to \infty$

1.25 $D(t) \to \infty$ as $t \to \infty$ (if the diameter of the universe is idealized to be infinitely large)

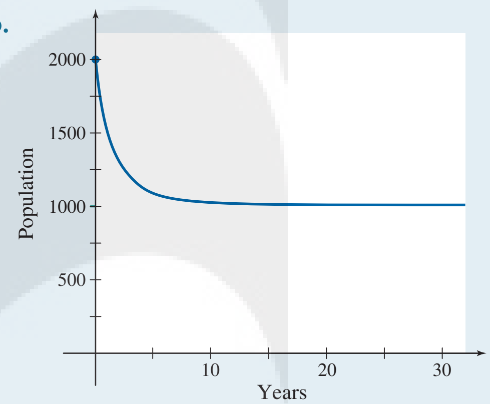
1.26 **a.** $C(x) \to N_0$ as $x \to \infty$

b. If the ratio of fish caught annually to fish dying annually is very large, then virtually all of the original number N_0 of fish are caught by fishing.

1.27 **a.**

t	*N*
0	2000
5	1092
10	1014
15	1002
20	1000
25	1000
30	1000

b.



c. The limit is 1000.

d. The population declines and stabilizes at around 1000 individuals.

EXERCISE SET 1.4

CHECK YOUR UNDERSTANDING

1. The limit as $x \to \infty$ shows which of the following?

a. The behavior of the graph at the right tail end

b. The behavior of the graph at the left tail end

c. The largest possible function value

d. None of the above

2. Is it possible to have a function that is always decreasing and such that the limit as $x \to \infty$ is ∞?

3. The symbol ∞ is which of the following?

a. The largest number there is

b. Not as big as $\infty + 1$

c. Not a number at all

d. None of the above

4. The limit as $x \to -\infty$ shows which of the following?

a. The negative of the limit as $x \to \infty$

b. The behavior of the graph at the left tail end

c. The same as the limit as $x \to \infty$

d. None of the above

5. If we want to know what the graph of $f(x)$ looks like for large x, we should calculate ____________.

6. True or false: A table of values can be useful in estimating a limit.

7. True or false: Every function has a limit as $x \to \infty$.

8. True or false: If a graph gets very close to a horizontal line, it can never move away from it.

9. True or false: If the denominator of a fraction is near 0, then the fraction must be very large.

10. If $f(x) \to 0$ as $x \to -\infty$, then the limit as $x \to -\infty$ of $-f(x)$ is ____________.

SKILL BUILDING

In Exercises 11 through 27, find the indicated limit.

11. $y = \dfrac{1}{x^2}$ as $x \to \infty$

12. $y = \dfrac{1}{x^2}$ as $x \to -\infty$

13. $y = -\dfrac{1}{x^2}$ as $x \to -\infty$

14. $y = -\dfrac{1}{x^2}$ as $x \to \infty$

15. $y = \dfrac{1}{x}$ as $x \to -\infty$

16. $y = 3 - \dfrac{1}{x^2}$ as $x \to \infty$

17. $y = 4 + \dfrac{1}{x}$ as $x \to -\infty$

18. $y = \sqrt{x}$ as $x \to \infty$

19. $y = \dfrac{1}{\sqrt{x}} - 3$ as $x \to \infty$

20. $y = x^2 + x + 1$ as $x \to \infty$

21. $y = -3x$ as $x \to \infty$

22. $y = \dfrac{1}{x+5}$ as $x \to \infty$

23. $y = \dfrac{3}{x} - 5$ as $x \to \infty$

24. $y = \dfrac{\frac{2}{x} + 3}{\frac{1}{x} + 1}$ as $x \to \infty$

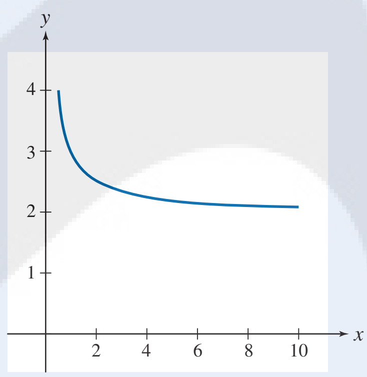
25. $y = \dfrac{2 + \frac{1}{x} + \frac{4}{x^2}}{4 - \frac{1}{x} - \frac{4}{x^2}}$ as $x \to \infty$

26. $y = \dfrac{9 - \frac{5}{x} + \frac{14}{\sqrt{x}}}{3 + \frac{1}{x}}$ as $x \to \infty$

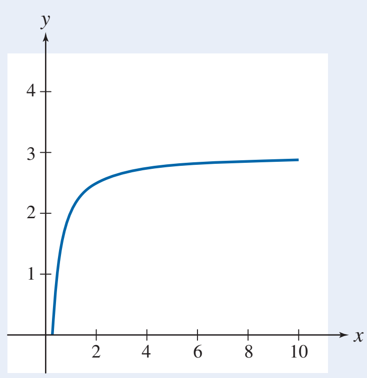
27. $y = \dfrac{4}{x} + 3$ as $x \to -\infty$

Estimating limits based on graphs. In Exercises 28 through 31, estimate the indicated limit of $f(x)$ based on the given graph of the function f.

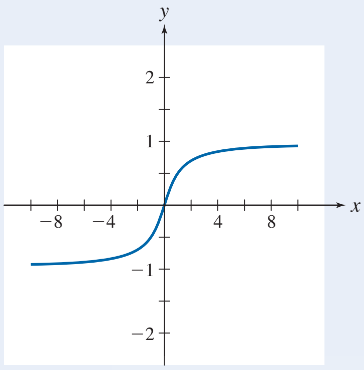
28. As $x \to \infty$



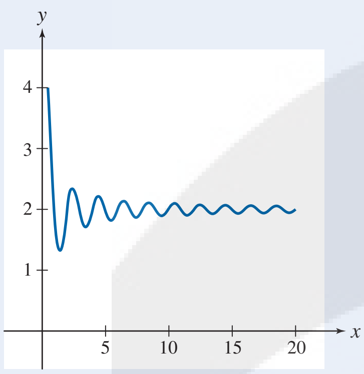
29. As $x \to \infty$



30. As $x \to \infty$ and as $x \to -\infty$



31. As $x \to \infty$



Estimating limits based on tables of values. In Exercises 32 through 35, use the given table of values for the function f to estimate the indicated limit of $f(x)$.

32. As $x \to \infty$

x	$f(x)$
100	1.35
200	1.79
300	1.87
400	1.96
500	1.98

33. As $x \to \infty$

x	$f(x)$
100	6.33
200	6.21
300	6.15
400	6.08
500	6.04

34. As $x \to \infty$

x	$f(x)$
100	3.91
200	3.76
300	3.68
400	3.57
500	3.53

35. As $x \to -\infty$

x	$f(x)$
–100	2.11
–200	2.38
–300	2.41
–400	2.47
–500	2.49

PROBLEMS

Sketching graphs. In Exercises 36 through 43, sketch a graph that satisfies the given condition. There is more than one correct answer to each of these problems. The answers provided are examples.

36. Concave down, and the limit as $x \to \infty$ is 3

37. Concave up, and the limit as $x \to -\infty$ is 2

38. The limit as $x \to \infty$ is 2, and the limit as $x \to -\infty$ is -2.

39. The limit as $x \to \infty$ is -1, and the limit as $x \to -\infty$ is ∞.

40. The limit as $x \to \infty$ is $-\infty$, and the limit as $x \to -\infty$ is ∞.

41. The limit as $x \to \infty$ is 4, and the limit as $x \to -\infty$ is 4.

42. Increasing, and the limit as $x \to \infty$ is 5

43. Decreasing, and the limit as $x \to \infty$ is 2

Piecemeal calculations. Suppose that the limit as $x \to \infty$ of $f(x)$ is 4. In Exercises 44 through 52, use this fact to calculate the limit as $x \to \infty$ of the given expression.

44. $f(x)+5$

45. $3f(x)$

46. $\dfrac{1}{f(x)}$

47. $\dfrac{f(x)+1}{f(x)+2}$

48. $\sqrt{f(x)}$

49. $(f(x))^2$

50. $1+3f(x)$

51. $\dfrac{25}{f(x)+1}$

52. $(2+f(x))^2$

Limits of functions of the form $y = \dfrac{ax+b}{cx+d}$. Exercises 53 through 59 deal with limits of certain ratios. In each case, find the limit as $x \to \infty$ of the given function.

53. $y = \dfrac{2x-4}{x+5}$

54. $y = \dfrac{x+1}{x-1}$

55. $y = \dfrac{6x+1}{2x+3}$

56. $y = \dfrac{3x+1}{x-4}$

57. $y = \dfrac{3}{x+5}$

58. $y = \dfrac{3x+4}{5}$

59. $y = \dfrac{1-x}{1+x}$

Using technology to estimate limits. For Exercises 60 through 66, use technology to produce a graph, table of values, or both to help you estimate the indicated limit.

60. Estimate the limit as $x \to \infty$ of $y = \left(1 + \dfrac{1}{x}\right)^x$.

61. Estimate the limit as $x \to \infty$ of $y = (1+x)^{1/x}$.

62. Estimate the limit as $x \to \infty$ of $y = x \times (2^{1/x} - 1)$.

63. Estimate the limit as $x \to \infty$ of $y = \sqrt{x^2 + 6x} - x$.

64. Estimate the limit as $x \to -\infty$ of $y = \dfrac{3^x - 1}{2^x - 1}$.

65. Estimate the limit as $x \to \infty$ of $y = \dfrac{12\sqrt{x}}{3 + 4\sqrt{x}}$.

66. Estimate the limit as $x \to \infty$ of $y = \dfrac{1}{x\left(\left(1+\frac{1}{x}\right)^2 - 1\right)}$.

MODELS AND APPLICATIONS

67. Carrying capacity. A breeding group of animals is introduced into a protected area. Figure 1.74 shows the graph of the population N as a function of the time t in years since the animals were introduced. The environmental carrying capacity for a species in a given region is the largest population of the species that the given environment can sustain. Use the graph to estimate the environmental carrying capacity.

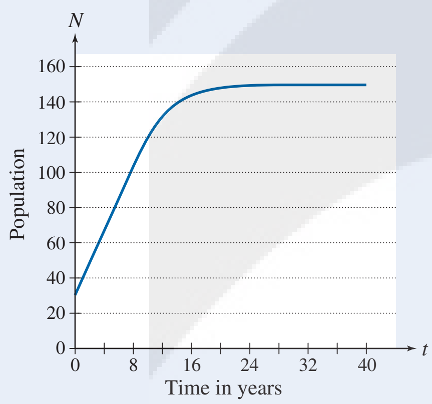


Figure 1.74 A breeding animal population

68. A box. The box shown in Figure 1.75 has a square base of width w feet and a height of h feet. Assume that the volume is 1 cubic foot.

a. Show that the height h of the cube is related to the width w of the base by $h(w) = \dfrac{1}{w^2}$.

b. We know from the formula in part a that $h(w) \to 0$ as $w \to \infty$. Explain what this means in terms of the height as the width increases.

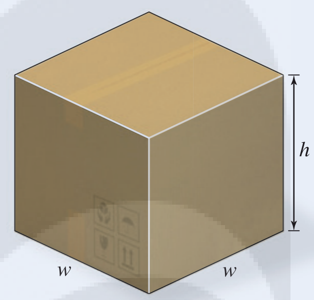


Figure 1.75 A box with a square base

69. An astronaut's view of Earth. The portion E of Earth's surface visible from a spacecraft depends on the height x above the surface of Earth. The relationship[8] is

$$E = \frac{x}{2R + 2x}$$

where R is the radius of Earth.

a. What is the limit as $x \to \infty$ of E?

b. Explain the meaning of the limit you calculated in part a in terms of the visible portion of Earth's surface.

70. Food consumption by sheep. The amount C of food consumed in a day by a sheep is a function of the amount V of vegetation available. The relationship is given by

$$C(V) = \frac{3V}{50 + V}$$

[8] *Space Mathematics,* B. Kastner, published by NASA in 1985.

Here C is measured in pounds and V in pounds per acre.

a. How much food will a sheep consume if there are 400 pounds of vegetation per acre?

b. What is the most a sheep will consume no matter what the vegetation level?

71. Ohm's law. Ohm's law says that when electric current is flowing across a resistor, the current i, measured in amperes, can be calculated from the voltage v, measured in volts, and the resistance R, measured in ohms. For a fixed voltage v, the current can be thought of as a function of the resistance R, and the relationship is

$$i(R) = \frac{v}{R}$$

What is the limit as $R \to \infty$ of $i(R)$? Explain what this limit means in terms of the current and resistance.

72. A cup of cocoa. The temperature C of a cup of cocoa t minutes after it is poured is given by

$$C = 125 \times 0.95^t + 75 \text{ degrees Fahrenheit}$$

a. Make a table of values showing the temperature of the cocoa in 20-minute intervals over the first 2 hours after it is poured.

b. Use a graphing utility to make the graph of C versus t over the first 2 hours of cooling.

c. Based on the table of values and the graph, estimate the value of the limit as $t \to \infty$ of C.

d. Explain the meaning of the limit from part c in terms of the temperature of the cocoa.

e. What is the temperature of the room in which the cocoa sits?

73. Baking a pie. A pie initially at room temperature is placed in a preheated oven to bake. Its temperature after t minutes in the oven is given by

$$P(t) = 400 - 325 \times 0.98^t \text{ degrees Fahrenheit}$$

a. Make a table of values that shows the temperature of the pie in 20-minute intervals over the first 2 hours of baking.

b. What is room temperature?

c. Use a graphing utility to make a graph of temperature versus time over the first four hours of baking.

d. Based on the graph, what is the temperature of the oven?

74. Bluebirds. Ecologists are tracking a bluebird population on a game preserve. They find that t years after observation began, the population N is given by

$$N = \frac{3450t + 1400}{10t + 7}$$

a. What was the bluebird population when tracking began?

b. According to the model, what eventual bluebird population can be expected?

75. Effective annual rate. If you borrow money, the lending institution must reveal the annual percentage rate, or APR. Interest is normally compounded, and the actual interest rate charged is called the effective annual rate, or EAR. The EAR depends on the number n of compounding periods per year. For example, $n = 1$ corresponds to yearly compounding, $n = 12$ to monthly compounding, and $n = 365$ to daily compounding. The relationship is given by

$$\text{EAR} = 100\left(\left(1 + \frac{\text{APR}}{100n}\right)^n - 1\right)$$

Here the APR and the EAR are percentages. Consider a loan with an APR of 8%.

a. Calculate the EAR in the case of yearly, monthly, and daily compounding. Report your answers correct to three decimal places.

b. Use a graphing utility to plot the graph of EAR versus APR.

c. Suppose the lending institution announced its intention to add another compounding period. Would the addition of a compounding period have more of an effect on the EAR in the case of monthly or of daily compounding?

d. What (approximately) is the maximum effective interest rate no matter how often interest is compounded? (Although not obvious, it is a fact that such a number exists.)

76. A snowball. Upon reentering the house on a winter day, your child deposits a dirty snowball on the carpet. Let $S(t)$ be the size of the snowball remaining after t minutes.

a. What is the limit as $t \to \infty$ of S?

b. Explain what has occurred when S is near its limiting value in terms of the size of the snowball.

77. Value of a dollar. Inflation I causes prices to increase. As inflation increases, the value of a dollar, in terms of what it will buy, decreases. The percentage decrease D is given by

$$D = \frac{100I}{100 + I}$$

Here I is a percentage.

a. Calculate $D(10)$, and explain in terms of the inflation rate and the value of the dollar the meaning of your calculation.

b. Calculate the limit as $I \to \infty$ of D, and explain the meaning of your calculation in terms of the inflation rate and the value of the dollar.

78. The term of a loan. If you borrow \$30,000 at an APR of 9% in order to buy a car, then your monthly payment M depends on the term t of the loan, which is the number of months required to repay the loan. The relationship is

$$M = \frac{30,000 \times 0.0075 \times 1.0075^t}{1.0075^t - 1} \text{ dollars}$$

a. Make a table of values for M using $t = 36, 48, 60, 200, 500$, and 1000.

b. Based on the table of values, estimate the value of the limit as $t \to \infty$ of M.

c. Explain the meaning of the number from part b in terms of the monthly payment.

d. One month's interest on \$30,000 is \$225. Explain the relationship of this number to the limit you calculated in part b.

CHALLENGE EXERCISES FOR INDIVIDUALS OR GROUPS

Calculating limits. For Exercises 79 through 84, assume that $f(x) \to 4$ as $x \to \infty$ and that $g(x) \to \infty$ as $x \to \infty$. Assume in addition that $f(x) \neq 4$ for all x and that $g(x)$ is never 0. You may find it necessary to perform some algebraic manipulations before you can calculate the indicated limit.

79. Calculate the limit as $x \to \infty$ of $\dfrac{(f(x))^2 - 16}{f(x) - 4}$.

80. Calculate the limit as $x \to \infty$ of $\dfrac{1}{|f(x) - 4|}$.

81. Calculate the limit as $x \to \infty$ of $\dfrac{f(x)}{g(x)}$.

82. Calculate the limit as $x \to \infty$ of $\dfrac{6g(x) + 5}{2g(x) + 9}$.

83. Calculate the limit as $x \to \infty$ of $f(x)g(x)$.

84. Calculate the limit as $x \to \infty$ of $f(g(x))$.

85. The squeeze theorem. Suppose that $f(x) \leq g(x) \leq h(x)$ for all x. Assume also that the limit of both $f(x)$ and $h(x)$ as $x \to \infty$ is 7. What is the limit as $x \to \infty$ of $g(x)$? Explain.

86. What is the limit? Figure 1.76 shows the graph of a function f. What does the graph suggest regarding the limit as $x \to \infty$ of $f(x)$?

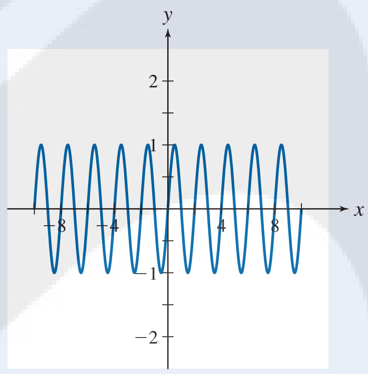


Figure 1.76 Possible limit as $x \to \infty$

REVIEW AND REFRESH: Exercises from Previous Sections

87. From Section P.1: Find the distance between (1, 4) and (2, −1).

Answer: $\sqrt{26}$

88. From Section P.2: Solve the inequality $3x - 1 < 7 - x$.

Answer: $x < 2$

89. From Section P.3: Solve the inequality $x^2 - 5x + 4 < 0$.

Answer: $1 < x < 4$

90. From Section 1.1: If $f(x) = \dfrac{x}{x+1}$, find $f(1-x)$.

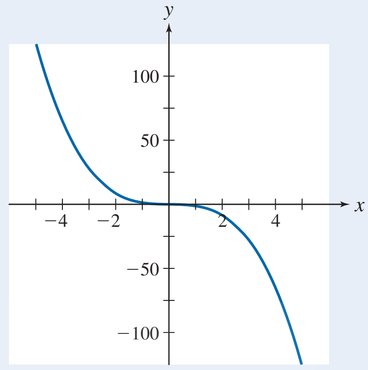
Answer: $\dfrac{1-x}{2-x}$

91. From Section 1.2: Calculate the average rate of change of $f(x) = 3x - 2$ from $x = 10$ to $x = 20$.

Answer: 3

92. From Section 1.3: Sketch the graph of a function that is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.

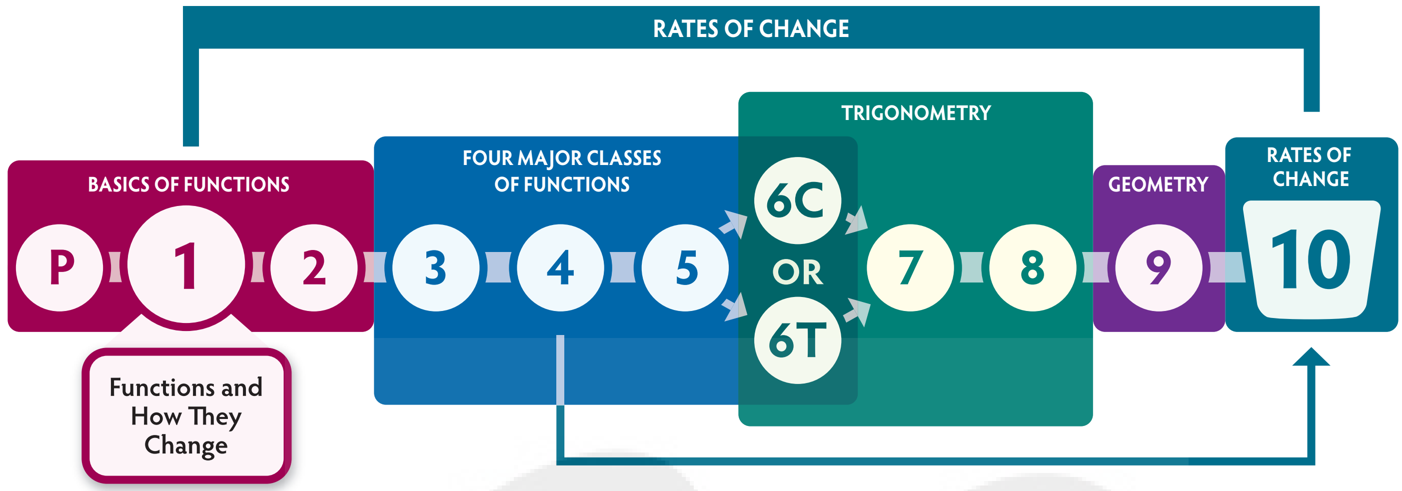
Answer:



93. From Section 1.3: **True or false:** If the function f has a local maximum at $x = 1$, then $f(1)$ is the largest value the function ever has.

Answer: **False.** It's the largest value taken near $x = 1$, but f might take larger values farther away from $x = 1$.

CHAPTER ROADMAP AND STUDY GUIDE



1.1 The Basics of Functions

Functions are fundamental tools for modeling real-world phenomena.

Definition of a Function

A function gives a rule for assigning some objects to others.

Evaluating Functions

Functions are evaluated by replacing the independent variable by the desired quantity.

Domain and Range

The domain comprises the allowable inputs for a function, and the range comprises the outputs.

Equations and Functions

An equation may determine a function.

Graphs and Functions

Graphs are visual representations of functions.

MODELS AND APPLICATIONS
Representing Costs of Everyday Purchases

1.2 Average Rates of Change

The average rate of change is one way to measure change over an interval.

Secant Lines

A secant line can be used to measure the rate of change of a function.

MODELS AND APPLICATIONS
Ants, Ebola, and Temperature

1.3 Graphs and Rates of Change

Rates of change are the key to a dynamic view of functions.

An Intuitive Description of Rates of Change

The rate of change is the slope of a line tangent to the graph.

Increasing and Decreasing Functions

Rates of change can show where functions are increasing and where they are decreasing.

Maximum and Minimum Values

Functions may reach maximum or minimum values where the rate of change is zero.

Concavity and Rates of Change

Concavity describes the shape of a graph and can be characterized in terms of rates of change.

MODELS AND APPLICATIONS

Sales Charts

1.4 Limits and End Behavior of Graphs

The limit at infinity describes the long-term behavior of functions.

An Intuitive Discussion of Limits at Infinity

The limit at infinity is understood in terms of the end behavior of the graph.

Examples of Limits at Infinity

A few cases illustrate common patterns for the end behavior of graphs.

Elementary Limit Calculations

The intuitive notion of limits is used to perform simple limit calculations.

Breaking Limit Calculations into Simpler Steps

Computing limits of complicated functions is accomplished in steps.

MODELS AND APPLICATIONS

Long-Term Behavior

CHAPTER QUIZ

1. If $f(x) = \frac{x-1}{x}$, calculate $f(2)$ and $f(x+1)$. XR **Example 1.1**

Answer: $f(2) = \frac{1}{2}$; $f(x+1) = \frac{x}{x+1}$

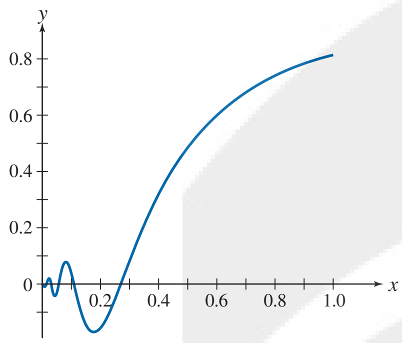
2. Find the domain and range of $f(x) = \frac{1}{\sqrt{x}}$. XR **Example 1.5**

Answer: Domain $(0, \infty)$. Range $(0, \infty)$.

3. Does the equation $\frac{1}{y} + x = 4$ determine y as a function of x? If so, what is its domain? XR **Example 1.6**

Answer: Yes. The domain is $(-\infty, 4) \cup (4, \infty)$.

4. Is this graph the graph of a function? XR **Example 1.7**



Answer: Yes

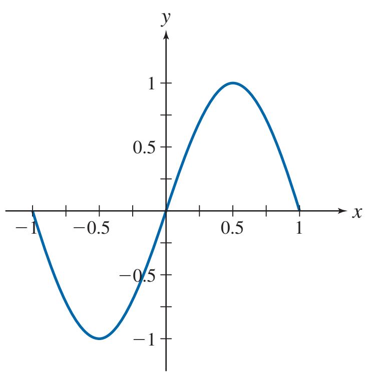
5. Calculate the average rate of change of $f(x) = x^2 + 3x + 1$ on the interval $[x, x+h]$. XR **Example 1.11**

Answer: $2x + h + 3$

6. A certain bird population N after t years is given by $N(t) = 250 \times 1.1^t$. Calculate the average rate of change in N from the third through the fifth years. (Round the result to one decimal place, and be sure to include proper units.) Explain the meaning of your answer in terms of the bird population. XR **Example 1.12**

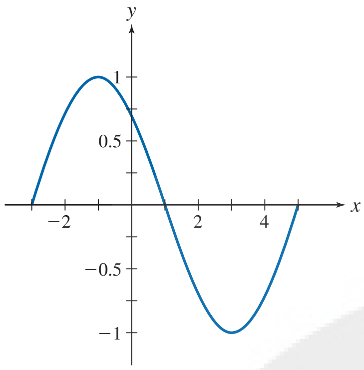
Answer: 34.9 birds per year. From the third through the fifth year, the bird population grew on average by 34.9 birds each year.

7. The graph of a function whose domain is $[-1, 1]$ is shown. Identify the regions of increase and decrease, locate the regions where the rate of change is positive and the regions where it is negative, and find all local maxima and minima and all absolute maxima and minima. XR **Example 1.17**



Answer: Increasing on $[-0.5, 0.5]$, decreasing on $[-1, -0.5]$ and on $[0.5, 1]$. Rate of change positive on $(-0.5, 0.5)$ and negative on $(-1, -0.5)$ and $(0.5, 1)$. Local and absolute minimum value of -1 at $x = -0.5$; local and absolute maximum value of 1 at $x = 0.5$.

8. The graph of a function is shown. Identify the intervals where the graph is concave up and where it is concave down. Also find any points of inflection. **XR Example 1.20**

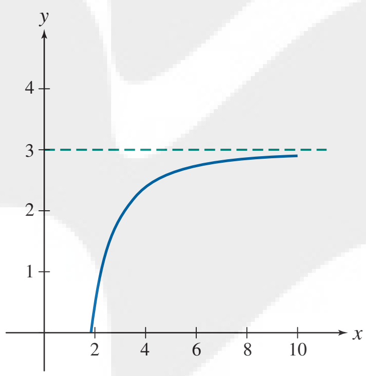


Answer: Concave down on $[-3, 1)$. Concave up on $(1, 5]$. Point of inflection at the point $(1, 0)$.

9. Find the limit as $x \to \infty$ of $y = 3 - \dfrac{10}{x^2}$. Also show a table of values and an appropriate graph. **XR Example 1.24**

Answer: $3 - \dfrac{10}{x^2} \to 3$ as $x \to \infty$.

x	$3 - \dfrac{10}{x^2}$
10	2.9
100	2.999
1000	2.99999



10. Find the limit as $x \to \infty$ of $y = \dfrac{2x-3}{3x+4}$. **XR Example 1.24**

Answer: $\dfrac{2}{3}$

CHAPTER REVIEW EXERCISES

Section 1.1

Calculating function values. In Exercises 1 through 11, a function is given. State the domain of the function, and then calculate the indicated function value.

1. $f(x) = \dfrac{x+1}{x-1}$. Calculate $f(3)$.

Answer: Domain $(-\infty, 1) \cup (1, \infty)$. $f(3) = 2$.

2. $f(x) = \dfrac{x}{4-x}$. Calculate $f(1-x)$.

Answer: Domain $(-\infty, 4) \cup (4, \infty)$. $f(1-x) = \dfrac{1-x}{x+3}$.

3. $f(x) = \sqrt{x+5}$. Calculate $f(x^2 - 5)$.

Answer: Domain $[-5, \infty)$. $f(x^2 - 5) = |x|$.

4. $f(x) = \dfrac{x+1}{\sqrt{x-3}}$. Calculate $f(7)$.

Answer: Domain $(3, \infty)$. $f(7) = 4$.

5. $g(x) = (x+1)^2 - (x-1)^2$. Calculate $g(t+1)$.

Answer: Domain $\mathbb{R}$. $g(t+1) = 4t + 4$.

6. $g(x) = x^2$. Calculate $\dfrac{g(x+h) - g(x)}{h}$, $h \neq 0$.

Answer: Domain $\mathbb{R}$. $\dfrac{g(x+h) - g(x)}{h} = 2x + h$.

7. $h(x) = \sqrt{x}$. Calculate $\dfrac{x-a}{h(x) - h(a)}$, $x \neq a$, $x, a > 0$.

Answer: Domain $[0, \infty)$. $\dfrac{x-a}{h(x) - h(a)} = \sqrt{x} + \sqrt{a}$.

8. $s(t) = \dfrac{t^2 - \pi^2 + 1 + 2t}{t - \pi + 1}$. Calculate $s(p-1)$, $p \neq \pi$.

Answer: Domain $(-\infty, \pi - 1) \cup (\pi - 1, \infty)$.
$s(p-1) = p + \pi$.

9. $k(x) = \dfrac{x}{x^2 - 5x + 6}$. Calculate $k\left(\dfrac{1}{x}\right)$, $x \neq 0$.

Answer: Domain $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.
$k\left(\dfrac{1}{x}\right) = \dfrac{x}{1 - 5x + 6x^2}$.

10. $g(t) = \dfrac{1}{2 - \sqrt{t}}$. Calculate $g(9)$.

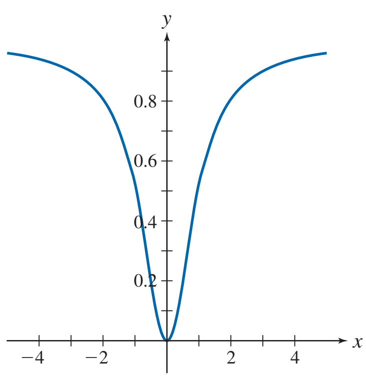
Answer: Domain $(0, 4) \cup (4, \infty)$. $g(9) = -1$.

11. $f(z) = \dfrac{z+4}{z - \sqrt{z}}$. Calculate $f(4)$.

Answer: Domain $(0, 1) \cup (1, \infty)$. $f(4) = 4$.

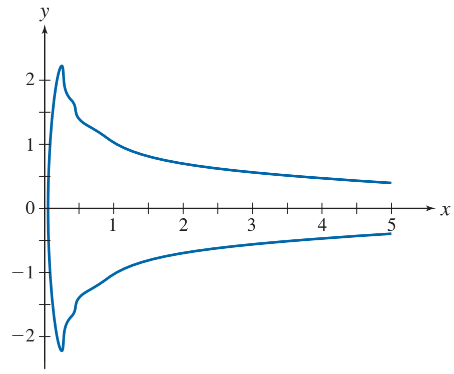
Is this the graph of a function?

12. Is the graph below the graph of a function? Explain your answer.



Answer: Yes, because it passes the vertical line test.

13. Is the graph below the graph of a function? Explain your answer.



Answer: No, because it fails the vertical line test.

Finding the range.

14. Find the range of $f(x) = x^2 + 2$.

Answer: $[2, \infty)$

15. Find the range of $u(t) = 3t - 8$.

Answer: $\mathbb{R} = (-\infty, \infty)$

Determining *y* as a function of *x*.

16. Does the equation $x = \dfrac{y-1}{y+1}$ determine y as a function of x?

Answer: Yes

17. Does the equation $y = \dfrac{x-1}{y-1}$ determine y as a function of x?

Answer: No

18. Does the equation $(y+3)^2 + (2-x)^2 = 0$ determine y as a function of x?

Answer: Yes

Section 1.2

Calculating average rates of change. In Exercises 19 through 27, calculate the average rate of change of the given function on the given interval.

19. $f(x) = x^2$. Interval $[2, 4]$.

Answer: 6

20. $f(x) = 3x - 4$. Interval $[3, 7]$.

Answer: 3

21. $f(x) = \dfrac{1}{x}$. Interval $[1, 3]$.

Answer: $-\dfrac{1}{3}$

22. $g(x) = \sqrt{x}$. Interval [9, 25].

Answer: $\frac{1}{8}$

23. $h(x) = \frac{x-2}{x+2}$. Interval [−1, 1].

Answer: $\frac{4}{3}$

24. $f(x) = x^2 + 1$. Interval $[a, b]$.

Answer: $a + b$

25. $u(t) = t^2 + t$. Interval $[x, x + h]$.

Answer: $2x + h + 1$

26. $f(x) = \frac{1}{x}$. Interval $[a, b]$, $a \neq 0, b \neq 0$.

Answer: $-\frac{1}{ab}$

27. $v(s) = \frac{s-1}{s+1}$. Interval $[1, x]$.

Answer: $\frac{1}{x+1}$

A flu outbreak.

28. There is a flu outbreak in a small town. By day 3, the cumulative number of flu cases reported is 30. By day 5, the cumulative number is 52. Calculate the average rate of change in the cumulative number of flu cases from day 3 to day 5, and explain what it means in terms of new cases. Then use the average rate of change to estimate the cumulative number of cases expected by day 6.

Answer: From day 5 to day 7, there were an average of 11 new cases per day. A cumulative number of 63 cases is expected by day 6.

Section 1.3

Find the rate of change.

29. The tangent line at $x = 2$ to the graph of $f(x) = x^3$ passes through the points $(2, f(2))$ and (3, 20). Find the rate of change of f at $x = 2$.

Answer: 12

Features of graphs. The graphs of the functions $f(x)$ and $g(x)$ are shown in **Figure 1.77** and **Figure 1.78**. In each case, the domain of the function corresponds to the extent of the graph. Exercises 30 through 41 refer to these functions.

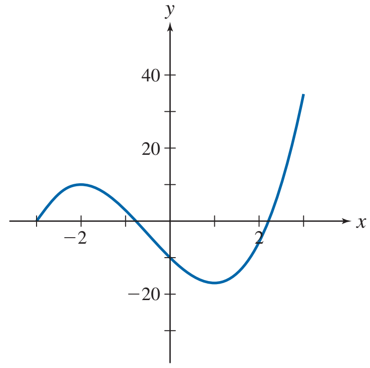


Figure 1.77 The graph of f

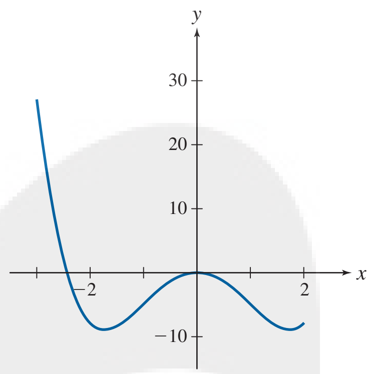


Figure 1.78 The graph of g

30. Identify all local maxima for the function f.

Answer: Local maximum value of 10 at $x = -2$

31. Give the x-value for all local minima of the function f.

Answer: $x = 1$

32. Identify the absolute maximum for f.

Answer: Absolute maximum value of 35 at $x = 3$

33. Over what regions is f increasing?

Answer: [−3, −2] and [1, 3]

34. Over what regions is f decreasing?

Answer: [−2, 1]

35. Over what regions is the rate of change of f positive?

Answer: (−3, −2) and (1, 3)

36. Over what regions is the rate of change of f negative?

Answer: (−2, 1)

37. Over what regions is the graph of g concave up?

Answer: [−3, −1) and (1, 2]

38. Over what regions is the graph of g concave down?

Answer: $(-1, 1)$

39. Identify all local maxima for g.

Answer: Local maximum value of 0 at $x = 0$

40. Identify all points of inflection for g.

Answer: $(-1, -5)$ and $(1, -5)$

41. How many absolute minima does g have?

Answer: Two

Determining concavity and decrease.

42. What is the concavity of the graph of $y = x^2$?

Answer: It is concave up.

43. Over what region is the graph of $y = x^2$ decreasing?

Answer: $(-\infty, 0]$

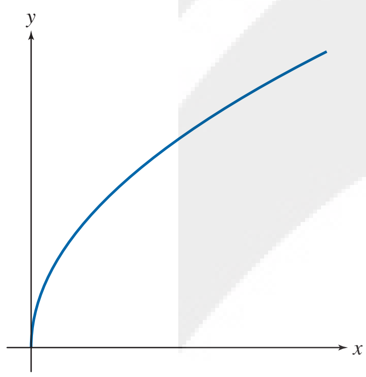
44. Determine the concavity of the graph of $f(x) = \frac{1}{x}$.

Answer: Concave down on $(-\infty, 0)$, concave up on $(0, \infty)$

Sketching and analyzing graphs.

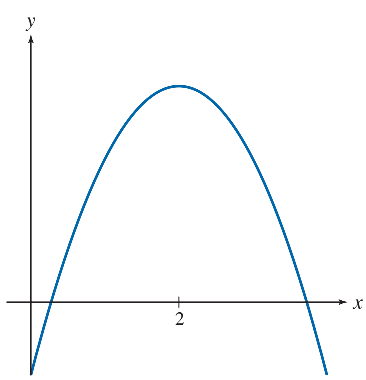
45. Sketch a graph that is increasing at a decreasing rate.

Answer:

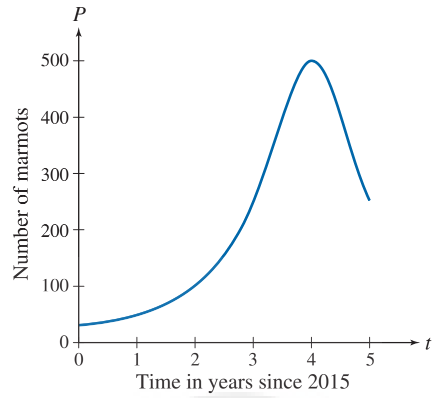


46. Sketch a graph that is concave down and has a local maximum at $x = 2$.

Answer:



47. The following graph shows the marmot population versus time in a restricted region. Over what period was the population increasing? What was the maximum number of marmots during the period shown by the graph?



Answer: The population increased from 2015 to 2019. The maximum number of marmots was 500.

Section 1.4

Calculating limits. In Exercises 48 through 59, calculate the indicated limit.

48. $y = \frac{1}{x}$ as $x \to \infty$

Answer: 0

49. $y = \frac{7}{x}$ as $x \to -\infty$

Answer: 0

50. $y = 3 - \frac{1}{x^2}$ as $x \to \infty$

Answer: 3

51. $y = x + \frac{1}{x}$ as $x \to \infty$

Answer: ∞

52. $y = \dfrac{6 + \frac{2}{x}}{3 - \frac{5}{x}}$ as $x \to -\infty$

Answer: 2

53. $y = \frac{100}{x + 5}$ as $x \to \infty$

Answer: 0

54. $y = x^3 + 1$ as $x \to -\infty$

Answer: $-\infty$

55. $y = \dfrac{12 - \frac{4}{x^2}}{3 + \frac{7}{x} - \frac{5}{x^2}}$ as $x \to \infty$

Answer: 4

56. $y = \dfrac{2}{3 + x^2}$ as $x \to -\infty$

Answer: 0

57. $y = \dfrac{3 + \frac{4}{x} - \frac{9}{x^2}}{1 + \frac{4}{x} + \frac{4}{x^2}}$ as $x \to \infty$

Answer: 3

58. $y = \dfrac{\frac{7}{x} + \frac{100}{x^2}}{1 + \frac{1}{x^3}}$ as $x \to -\infty$

Answer: 0

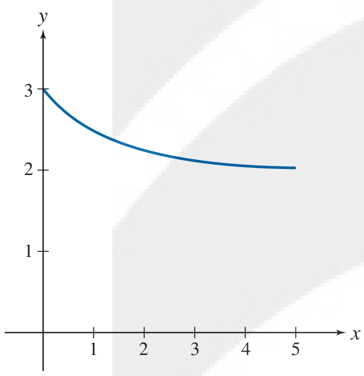
59. $y = \dfrac{30x + 2}{1 - 15x}$ as $x \to \infty$

Answer: -2

Sketching graphs. In Exercises 60 through 63, sketch a graph of a function f with the given properties. In each case, there is more than one correct answer. The answer provided is only one example.

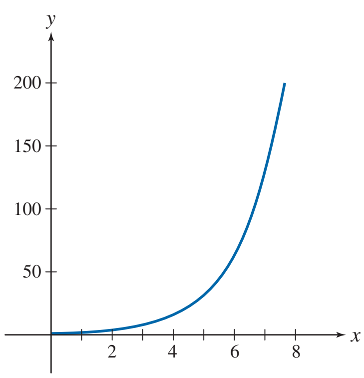
60. The function f is decreasing, and the limit as $x \to \infty$ is 2.

Answer:



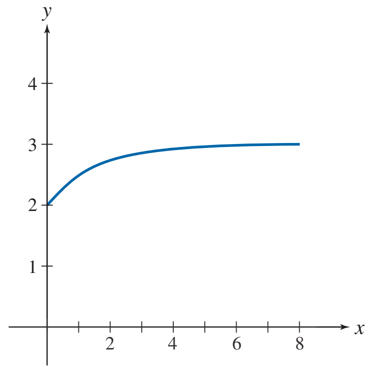
61. The limit as $x \to \infty$ is ∞.

Answer:



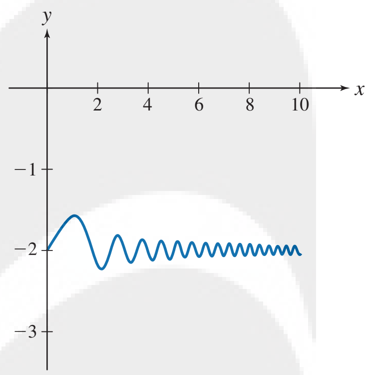
62. The graph is concave down, and the limit as $x \to \infty$ is 3.

Answer:



63. The function f is neither decreasing nor increasing, and the limit as $x \to \infty$ is -2.

Answer:



Food consumption.

64. The amount C of food consumed in a day by a sheep depends on the amount V of vegetation available. In a certain setting, this relationship is given by

$$C(V) = \frac{6V}{25 + 2V}$$

where C is measured in pounds, and V is measured in pounds per acre. We want to know the maximum amount of food a sheep will eat in a day no matter how much vegetation is available. Express your answer as a limit, and calculate its value.

Answer: The limit as $V \to \infty$ of $C = \dfrac{6V}{25 + 2V}$, which is 3 pounds.